

ALAIN BADIOU'S MISTAKE— TWO POSTULATES OF DIALECTIC MATERIALISM

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ABSTRACT. I discuss how Alain Badiou's *Logics of Worlds* attempts to rephrase his material dialectic philosophical project in terms of topos theory. It turns out that his account restricts to the so called local topos theory. In particular, his claim that categorical change is not genuine is based on a constrained understanding of topos theory. We then discuss his own 'postulate of materialism' and demonstrate that it has two different interpretations depending on whether it is articulated in local or elementary topos theory. While the main concerns in this paper are technical, we also address the serious consequences of topos theory that weigh Badiou's philosophical project.

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INTRODUCTION

Alain Badiou, perhaps the most prominent French philosopher today, is one of the key figures in *dialectic materialism*—a discourse that has overshadowed the past century of continental philosophy after the influence of radical social critique. But it is an interesting discourse to contrast also with contemporary scientific materialism. Badiou's philosophy, which touches several key themes in contemporary mathematics, can be interpreted as an endeavour in this direction.

Topos theory then plays a pivotal role to his argument. This is not so much because of the exact way in which topos theory came to geometrize and express categorically a new foundation to formal logic and set theory. Instead, while arguing that mathematics 'is ontology'¹, he discovers Paul Cohen's work in the incompleteness of set theory—a precursor and a prelude to the categorical structural approach to the 'topos' of mathematics. But still relying on set theory, Cohen's work is the only discourse of the 'subject' of mathematics that Badiou does respect.

This results with an interesting ambivalence. First, even if implicitly, the most elementary structures of topos theory are already immersed in Paul Cohen's² argument. Therefore, we can say that a 'topos' plays a pivotal role to the first one of Badiou's two great treatises: *Being and Event*. Lawvere and Tierney then introduced topos theory in the 1970's by making those structures, which are hiddenly immersed in Badiou's first argument, explicit. This resulted with something that goes beyond Cohen's original field of inquiry: an entirely new way to view mathematics.

By contrast, Badiou's³ latter work *Logics of Worlds* can be viewed as an attempt to support his own, still set-theoretic stand point over mathematical 'being'. Therefore, the precise beauty of his first argument—Cohen's work as a prelude to change—is condemned by the second treatise. He must have discovered topos theory because of this genealogical connection that then inspired him to write the second treatise in the first place. Even so, and because of this connection, he only discovers a literature that restricts to local theory: something reconcilable from the point of view of set theory. Choosing to delimit his discussion to this limited appearance is not as such a failure, of course: Cohen's argument can indeed be expressed as a local topos. However, this in no way undermines the precursors to change that were already immersed in Cohen's intuition. Badiou, by contrast, seems obliged to make a case against and to denounce the relevance of categorical techniques which could make that prelude articulate. Therefore, he says, the 'same mathematics' still continues; that Grothendieck's geometric insight incorrigibly fails as a genuine event; that there is no change in the 'way of doing mathematics'⁴.

This reductive, 'ontological' decision is constitutive from the beginning of his second argument, and any conclusions he draws on the materiality of phenomenology are to be read with caution. The basic line of thought comes down to *affirming the consequent*, that is, by showing that some categories take a particular form while then suggesting that all of them should do so.

In contrast, as we demonstrate, should we follow topos theory proper there are two materially (mathematically) distinct conditions that only locally—i.e. from the point of view of *local* topos theory, appear to reflect the same 'postulate of materialism' which is pivotal to Badiou's phenomenological argument. We can say that a local topos is *materially local* in relation to set theory, whereas a more general one (the so called Grothendieck-topos) satisfies conditions that make it *material* over *Sets* but not in the constrained, 'local' way. Faithful to Badiou's own vocabulary, we call the former, stronger condition an *atomic* version of the postulate of materialism, whereas a Grothendieck-topos satisfies only a *weak* version of the postulate. The weak version is not logically bounded or constitutive: this suggests that 'Grothendieck' does make a material difference

Of course, Alexander Grothendieck himself has little to do with the specific definition of a Grothendieck-topos, even if his insight stands behind them at least in a simplistic way. Namely, such material, *elementary* topoi form the first but not the last step in generalizing mathematics beyond set theory. They are particularly important, however, because they already articulate change in the way of doing mathematics in a way that can be materially circumscribed from the point of view of set theory. Grothendieck-topoi thus interestingly contrast with Badiou's own, ahistorical and 'pure' situation of mathematical ontology.

To understand the difference between the two doctrines of materialism, we need a categorical point of view—an insight absent in Badiou's own work. Category theory, even if debatably, can be considered as an alternative foundation to mathematics. In particular, if set theory is based on the formal dialect of logic and, in that respect, can be considered as a 'dialectical' one, category theory in contrast is based on *structural arguments* and applies diagrams that have more geometric meaning. In the context of post-structuralism—following Deleuze and Foucault—it is intriguing to treat the categorical approach as a 'diagrammatic' one as opposed to the 'dialectical' reason behind set theory. While topos theory then combines the two foundations by treating 'dialectics' itself 'diagrammatically', the focus shifts *from syntax to semantics*: to the question of the *meaning* of that what Badiou treats as the self-evident discourse of 'being'. In particular, it teaches us that the problem of *materiality itself is not inevitably a logically articulable question*.

As a brief historical overview, category theory was first introduced by Samuel Eilenberg and Saunders Mac Lane during the 1940's. In result, various fields of mathematics including algebraic geometry and geometric representation theory were revolutionarized. Philosophers have paid little attention to this shift⁵, however, and in particular in the *structuralist* shift in the focus of mathematics⁶.

Rather than standing itself as its new foundation, topos theory stands at the cross-road of the two, dialectic and categorical paradigms through which it intermediates the effects of change. To borrow Badiou's own, event-philosophical vocabulary, if category theory counts as an event, topos theory deals with the local consequences that are then visible also to set theory. It is not all clear, however, whether the categorical

event was the cause behind this localization or whether it was actually in set theory that something interesting did happen (as demonstrated by Cohen). From the point of view of his own argument, it would be meaningless to ask whether one of the two perspectives, 'top-down' and 'bottom-up' came first. As Badiou⁷ himself knows, the two always occur in tandem.

To introduce this event from below—on the the local side of set theory—Cohen did shows that there are sentences that are necessarily undecidable: there are contrary, transitive denumerable models of set-theory. We may choose one which verifies a certain statement while another one disproves it. Lacking categorical insight, however, neither Cohen nor Badiou could figure out how to think of such contradicting models or situations *together*. Badiou's philosophy could only deal with the occurrence of the 'inconsistent' as a 'generic' decision, that is, as a choice of only one among all situations $S(\wp)$. He thus believes that *one* needs to choose which context to inhabit instead of residing in and between many of them *all* at once. Topos theory, by contrast, does use categorical techniques to specifically express the amalgam of such situations so that the need to decide does not arise but possibly afterwards. Badiou's decision between situations then only emerges as a (local) projection of that topos onto set theory. Such projections (indicated by \wp) are in fact specific kind of geometric morphisms that topos theorists refer to as a 'points'. It is not by accident that a 'decision' in Badiou's vocabulary then stands for that precise procedure through which category theorists 'make a point'. There is much synergy between dialectical and scientific modes of materialisms.

My question then concerns how precisely to correct the way in which Badiou treats his point—the 'postulate of materialism'—and how to express it in a categorically adequate way. Ironically, Badiou⁸ himself claims that '[i]f one is willing to bolster one's confidence in the mathematics of objectivity, it is possible to take even further the thinking of the logico-ontological, of the chiasmus between the mathematics of being and the logic of appearing'. He further quotes Jean Dieudonné by saying that to earn the 'right' to speak 'one must master the active, modern mathematical corpus'⁹. Failure to do so leads to several philosophical issues that we also briefly discuss in the end of this paper. Sections 1–5 focus on Badiou's own formalism. Sections 6–11 shift

to categorical setting and express a correct version of several Badiou's statements. We mainly follow the works by Peter Johnstone¹⁰ and Saunders MacLane and Ieke Moerdijk¹¹.

1. FROM ONTOLOGY TO PHENOMENOLOGY

Since at least Kant's¹² *Pure Reason*, the question of causation has been pivotal to theoretical philosophy. What are the *a priori* rules of deduction that can be assumed without the use of senses? As if requiring no historical insight, Badiou¹³ assumes that the question of the 'pure' then best correlates with formal logic and set theory: that 'mathematics *is* ontology—the science of being qua being'—even despite the dialectic 'impasse of logic' (eg. Gödel, Tarski and Cohen).

When it comes to ontology, Badiou claims, an event can only be localized against this language; there is no other way for it to deploy its consequences but through those impasses which could not be communicated without the use of set-theoretic language. In this sense, even if the event is about what mathematics is not, his reductive understanding of mathematics becomes crucial to understanding precisely that what that 'what is not' *is*. Expressed in formal language, Badiou then deploys the event as a self-belonging, auto-affirmative multiple $e \in e$. It is the crux of his argument that the very existence of such an 'event-multiple' obstructs the axiom of foundation¹⁴.

1.1. **Axiom** (Foundation). For each non-empty set x there is an element $y \in x$ so that their intersection $x \cap y = \emptyset$ is empty.

The connection between the event and Cohen's argument, however, is rather 'metaphorical' and there is no mathematically material way to combine the two insights that occupy Badiou in the beginning and the end of the *Being and Event*, that is, the association between the event-multiple e and the 'generic' decision or set φ , which stands out as something 'supernumerary' to a particular transitive, denumerable model of set theory S . By contrast, the *Logics of Worlds* begins from a different set of problematics: he seeks to treat the question of 'phenomenology' instead. One might wonder if the latter work then connects the two approaches to the 'inconsistent', real being. Unfortunately, Badiou¹⁵ confesses that this is not the case.

But what precisely *happened* when Cohen used his insight to introduce the contradicting situations whose synthesis could not be but 'inconsistent' in the sense of classical set theory? We answer this question from the point of view of categorical techniques. They form another way to localize the consequences of the event. This coheres with the more topological insight according to which what is crucial to an event is not the way it is regulated in itself but as it happens to itself instead: the event 'makes the difference: not in space and time, but to space and time'¹⁶. Space and time are Kant's *a priori* categories for the 'trans-phenomenal real'. And if there is an event, it should happen to them.

This insight obviously lacks in Badiou's work, not least because the absence of categorical approach to his 'transitory ontology' as pointed out by Norman Madarasz¹⁷. The only topoi Badiou then deals with are so called *local topoi*: they are local in respect to set-theory and, in that restricted domain, Badiou is right in that phenomenology could only arise as a 'calculated phenomenology'.

2. CATEGORY THEORY—A NEW ADVENTURE?

The lack of categorical insight, however, does not mean the lack of categorical references. The categorical revolution is immersed in Badiou's own approach even if in a dead, unspeaking way. Indeed, by arguing that the formalism of the *Logics of Worlds* 'is very different from the one found in the *Being and Event* as it shifts from 'onto-logy' to 'onto-logy' Badiou refers to category theory¹⁸. It deals with a different formalism, he argues: an alternative approach to the 'trans-phenomenal real'¹⁹. Yet, he argues, in this precise sense category theory makes no difference; it has no implications to space and time as concepts.

If this only were the case. Instead, Badiou himself refuses to make a Kantian shift to understand a mathematical object in the categorical way: not in relation to what it consists of but in the way in which it relates to other objects instead. He cannot comply with what he himself regards as Kant's 'sanctimonious declaration that we can have no knowledge of this or that'²⁰ as such an inquiry would be 'always threatening you with detention, the authorization to platonize'²¹. In the sense that categorical objects have no contents, they are all empty. Yet Badiou fails to grasp the contents of incorporeality as it resides

even between such empty objects, that is, the statement ' $x \in A$ ' as an incorporeal *relationship*. Categorical topos theory, by contrast, treats it precisely as such: as an arrow between objects instead of assuming x and A to exist as corporeal entities like Badiou does.

By contrast, to define²² a category \mathcal{C} it consists of a collection of objects $\text{Ob}(\mathcal{C})$ and a class (possibly not a set in the ZFC axiomatics) morphisms or arrows $\text{Hom}(A, B)$ for any two objects $A, B \in \text{Ob}(\mathcal{C})$. In a topos, the predicate ' \in ' is viewed precisely as such an arrow. Therefore, objects are contained by a category but *objects themselves own no elements*. They have other ways to individuate: the class of relationships specified by Hom .

A category, not an object, is thus the corporeal entity for mathematics to incorporate. Much of Badiou's argument by contrast focuses still on what objects, not their category, do incorporate. But in the 'modern mathematical corpus'²³, which Badiou is unable to master, categorical thinking then shifts *from functionality to functoriality*: a (covariant) functor between categories $F : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ is a suitable set of maps $F : \text{Ob}(\mathcal{C}_1) \rightarrow \text{Ob}(\mathcal{C}_2)$ and $F : \text{Hom}(A, B) \rightarrow \text{Hom}(F(A), F(B))$. Not only does it transform objects to others but also the relationships between them in a 'diagrammatically' compatible way. It is another shortcoming in Badiou's work that he cannot distinguish between 'functions' and 'functors' and, ultimately, understand the role of diagrammatic argumentation.

Against this background, concepts like topology and sets can only be introduced afterwards, not as *a priori* entities but as they instead individuate from the class of objects and relationships—and functors between them. This means to say that they will be defined *functorially*. For example, a *point* in topos theory is a pair of functors $\mathcal{S}ets \rightarrow \mathcal{E}$. These two are categories of specific kind which are called elementary topoi. By contrast, Badiou himself treats topoi only as if they were themselves sets (as $\text{Ob}(\mathcal{E})$ instead of \mathcal{E}). The functorial idea that not only objects but also relationships do transform in a functorially meaningful way is ignored.

These are the basic hypothesis of my paper. Let us now demonstrate them in a mathematically adequate way.

3. BADIOU'S ONTOLOGICAL REDUCTION

The technical part of the argument now begins by introducing the local formalism that constitutes the basis of Badiou's own, 'calculated phenomenology'. Badiou is unwilling to give up his thesis that the history of thinking of being (ontology) is the history of mathematics and, as he reads it, that of *set theory*. It is then no accident that set theory is the regulatory framework under which topos theory is being expressed. He does not refer to topoi explicitly but rather to the so called complete Heyting algebras which are their procedural equivalents. However, he fails to mention that there are both 'internal' and 'external' Heyting algebras, the latter group of which refers to local topos theory, while it appears that he only discusses the latter—a reduction that guarantees that indeed that the categorical insight may give nothing new.

Indeed, the *external* complete Heyting algebras T then form a category of the so called T -sets²⁴, which are the basic objects in the 'world' of the *Logics of Worlds*. They local topoi or the so called 'locales' that are also 'sets' in the traditional sense of set theory. This 'constitution' of his worlds thus relies only upon *Badiou's own decision* to work on this particular regime of objects, even if that regime then becomes pivotal to his argument which seeks to denounce the relevance of category theory.

This problematic is particularly visible in the designation of the world \mathbf{m} (mathematically a topos) as a 'complete' (presentative) situation of being of '*universe* [which is] the (empty) concept of a being of the Whole'²⁵. He recognises the 'imposturous' nature of such a 'whole' in terms of Russell's paradox, but in actual mathematical practice the 'whole' \mathbf{m} becomes to signify the category of *Sets*²⁶—or any similar topos that localizable in terms of set theory. The vocabulary is somewhat confusing, however, because sometimes T is called the 'transcendental of the world', as if \mathbf{m} were defined only as a particular locale, while elsewhere \mathbf{m} refers to the category of all locales (*Loc*).

3.1. Definition. An external Heyting algebra is a set T with a partial order relation $<$, a minimal element $\mu \in T$, a maximal element $M \in T$. It further has a 'conjunction' operator $\wedge : T \times T \rightarrow T$ so that $p \wedge q \leq p$ and $p \wedge q = q \wedge p$. Furthermore, there is a *proposition* entailing the

equivalence $p \leq q$ if and only if $p \wedge q = p$. Furthermore $p \wedge M = p$ and $\mu \wedge p = \mu$ for any $p \in T$.

3.2. *Remark.* In the 'diagrammatic' language that pertains to categorical topoi, by contrast, the minimal and maximal elements of the lattice Ω can only be presented as diagrams, not as sets. The internal order relation \leq_Ω can then be defined as the so called *equaliser* of the conjunction \wedge and projection-map

$$\leq_\Omega \xrightarrow{e} \Omega \times \Omega \xrightarrow[\pi_1]{\wedge} L.$$

The symmetry can be expressed diagrammatically by saying that

$$\begin{array}{ccc} \leq_\Omega \cap \geq_\Omega & \xrightarrow{\quad} & \leq_\Omega \\ \downarrow & \searrow \Delta_{\circ\iota} & \downarrow e \\ \geq_\Omega & \xrightarrow{\quad} & \Omega \times \Omega \end{array}$$

is a pull-back and commutes. The minimal and maximal elements, in categorical language, refer to the elements evoked by the so called *initial* and *terminal* objects 0 and 1.

In the case of *local Grothendieck-topoi*—Grothendieck-topoi that support generators—the external Heyting algebra T emerges as a push-forward of the internal algebra Ω , the logic of the external algebra $T := \gamma_*(\Omega)$ is an analogous push-forward of the internal logic of Ω but this is not the case in general.

What Badiou further requires of this 'transcendental algebra' T is that it is *complete* as a Heyting algebra.

3.3. **Definition.** A *complete* external Heyting algebra T is an external Heyting algebra together with a function $\Sigma : \mathbf{P}T \rightarrow T$ (the least upper boundary) which is distributive with respect to \wedge . Formally this means that $\Sigma A \wedge b = \Sigma\{a \wedge b \mid a \in A\}$.

3.4. *Remark.* In terms of the subobject classifier Ω , the envelope can be defined as the map $\Omega^t : \Omega^\Omega \rightarrow \Omega^1 \cong \Omega$, which is internally left adjoint to the map $\downarrow \text{seg} : \Omega \rightarrow \Omega^\Omega$ that takes $p \in \Omega$ to the characteristic map of $\downarrow(p) = \{q \in \Omega \mid q \leq p\}$ ²⁷.

The importance the external complete Heyting algebra plays in the intuitionist logic relates to the fact that one may now define precisely such an intuitionist logic on the basis of the operations defined above.

3.5. **Definition** (Deduction). The dependence relation \Rightarrow is an operator satisfying

$$p \Rightarrow q = \Sigma\{t \mid p \cap t \leq q\}.$$

3.6. **Definition** (Negation). A negation $\neg : T \rightarrow T$ is a function so that

$$\neg p = \Sigma\{q \mid p \cap q = \mu\},$$

and it then satisfies $p \wedge \neg p = \mu$.

Unlike in what Badiou²⁸ calls a 'classical world' (usually called a Boolean topos, where $\neg\neg = 1_\Omega$), the negation \neg does not have to be reversible in general. In the domain of local topoi, this is only the case when the so called *internal axiom of choice*²⁹ is valid, that is, when epimorphisms split—for example in the case of set theory. However, one always has $p \leq \neg\neg p$. On the other hand, all Grothendieck-topoi—topoi still *materially* presentable over $\mathcal{S}ets$ —are possible to represent as parts of a Boolean topos.

4. ATOMIC OBJECTS

Badiou criticises the proper form of intuition associated with multiplicities such as space and time. However, his own 'intuitions' are constrained by set theory. His intuition is therefore as 'transitory' as is the ontology in terms of which it is expressed. Following this constrained line of reasoning, however, let me now discuss how Badiou encounters the question of 'atoms' and materiality: in terms of the so called 'atomic' T -sets.

If topos theory designates the subobject-classifier Ω relationally, the external, set-theoretic T -form reduces the classificatory question again into the incorporeal framework. There is a set-theoretical, explicit order-structure $(T, <)$ contra the more abstract relation $1 \rightarrow \Omega$ pertinent to categorical topos theory. Atoms then appear in terms of this operator $<$: the 'transcendental grading' that provides the 'unity through which all the manifold given in an intuition is united in a concept of the object'³⁰.

Formally, in terms of an external Heyting algebra this comes down to an entity (A, \mathbf{Id}) where A is a set and $\mathbf{Id} : A \rightarrow T$ is a function satisfying specific conditions.

4.1. **Definition** (Equaliser). First, there is an 'equaliser' to which Badiou refers as the 'identity' $\mathbf{Id} : A \times A \rightarrow T$ satisfies two conditions:

- 1) symmetry: $\mathbf{Id}(x, y) = \mathbf{Id}(y, x)$ and
- 2) transitivity: $\mathbf{Id}(x, y) \wedge \mathbf{Id}(y, z) \leq \mathbf{Id}(x, z)$.

They guarantee that the resulting 'quasi-object' is objective in the sense of being distinguished from the gaze of the 'subject': 'the differences in degree of appearance are not prescribed by the exteriority of the gaze'³¹.

This analogous 'identity'-function actually relates to the structural *equalization*-procedure as appears in category theory. Identities can be structurally understood as equivalence-relations. Given two arrows $X \rightrightarrows Y$, an *equaliser* (which always exists in a topos, given the existence of the subobject classifier Ω) is an object $Z \rightarrow X$ such that both induced maps $Z \rightarrow Y$ are the same. Given a topos-theoretic object X and U , pairs of elements of X over U can be compared or 'equalized' by a morphism $X^U \times X^U \xrightarrow{eq} \Omega^U$ structurally 'internalising' the synthetic notion of 'equality' between two U -elements.³² Now it is possible to formulate the cumbersome notion of the 'atom of appearing'.

4.2. **Definition.** An *atom* is a function $a : A \rightarrow T$ defined on a T -set (A, \mathbf{Id}) so that

- (A1) $a(x) \wedge \mathbf{Id}(x, y) \leq a(y)$ and
- (A2) $a(x) \wedge a(y) \leq \mathbf{Id}(x, y)$.

As expressed in Badiou's own vocabulary, an atom can be defined as an '*object-component which, intuitively, has at most one element in the following sense: if there is an element of A about which it can be said that it belongs absolutely to the component, then there is only one. This means that every other element that belongs to the component absolutely is identical, within appearing, to the first*'³³.

These two properties in the definition of an atom is highly motivated by the theory of T -sets (or Ω -sets in the standard terminology of topological logic). A map $A \rightarrow T$ satisfying the first inequality is usually thought as a 'subobject' of A , or formally a T -subset of A . The idea is that, given a T -subset $B \subset A$, we can consider the function

$$\mathbf{Id}_B(x) := a(x) = \Sigma\{\mathbf{Id}(x, y) \mid y \in B\}$$

and it is easy to verify that the first condition is satisfied. In the opposite direction, for a map a satisfying the first condition, the subset

$$B = \{x \mid a(x) = \mathbf{E}x := \mathbf{Id}(x, x)\}$$

is clearly a T -subset of A .

The second condition states that the subobject $a : A \rightarrow T$ is a *singleton*. This concept stems from the topos-theoretic internalization of the singleton-function $\{\cdot\} : a \mapsto \{a\}$ which determines a particular class of T -subsets of A that correspond to the atomic T -subsets. For example, in the case of an ordinary set S and an element $s \in S$ the singleton $\{s\} \subset S$ is a particular, atomic type of subset of S .

The question of 'elements' incorporated by an object can thus be expressed externally in Badiou's local theory but 'internally' in any elementary topos. For the same reason, there are two ways for an element to be 'atomic': in the first sense an 'element depends solely on the pure (mathematical) thinking of the multiple', whereas the second sense relates it 'to its transcendental indexing'³⁴. In topos theory, the distinction is slightly more cumbersome³⁵.

Badiou still requires a further definition in order to state the 'postulate of materialism'.

4.3. Definition. An atom $a : A \rightarrow T$ is *real* if there exists an element $x \in T$ so that $a(y) = \mathbf{Id}(x, y)$ for all $y \in A$.

This definition gives rise to the postulate inherent to Badiou's understanding of 'democratic materialism'.

4.4. Postulate (Postulate of Materialism). In a T -set (A, \mathbf{Id}) , every atom of appearance is real.

What the postulate designates is that there really needs to *exist* $s \in A$ for every suitable subset that structurally (read categorically) *appears* to serve same relations as the singleton $\{s\}$. In other words, what 'appears' materially, according to the postulate, has to 'be' in the set-theoretic, incorporeal sense of 'ontology'. Topos theoretically this formulation relates to the so called *axiom of support generators* (SG), which states that the terminal object 1 of the underlying topos is a generator. This means that the so called global elements, elements of the form $1 \rightarrow X$, are enough to determine any particular object X . Thus, it is this specific condition (support generators) that is assumed

by Badiou's notion of the 'unity' or 'constitution' of 'objects'. In particular this makes him cross the line—the one that Kant drew when he asked *Quid juris?* or 'Haven't you crossed the limit?' as Badiou³⁶ translates.

But even without assuming the postulate itself, that is, when considering a weaker category of T -sets not required to fulfill the postulate of atomism, the category of quasi- T -sets has a functor taking any quasi- T -set A into the corresponding quasi- T -set of singletons SA by $x \mapsto \{x\}$, where $SA \subset \mathbf{P}A$ and $\mathbf{P}A$ is the quasi- T -set of all quasi- T -subsets, that is, all maps $T \rightarrow A$ satisfying the first one of the two conditions of an atom designated by Badiou. It can then be shown that, in fact, SA itself is a *sheaf* whose all atoms are 'real' and which then is a proper T -set satisfying the 'postulate of materialism'. In fact, the category of T -*Sets* is equivalent to the category of T -sheaves $\mathcal{S}hvs(T, J)$ ³⁷. In the language of T -sets, the 'postulate of materialism' thus comes down to designating an equality between A and its completed set of singletons SA . We demonstrate this in the next section.

The particular objects Badiou discusses can now be defined as such quasi- T -sets whose all atoms are real; they give rise to what Badiou phrases as the 'ontological category par excellence'³⁸.

4.5. Definition. An object in the category of T -*Sets* is a pair (A, \mathbf{Id}) satisfying the above conditions so that every atom $a : A \rightarrow T$ is real.

Next, though not specifying this in the original text, Badiou attempts to show that such 'objects' indeed give rise to a mathematical category of T -*Sets*.

5. BADIOU'S 'SUBTLE SCHOLIUM': T -SETS ARE 'SHEAVES'

By following established accounts³⁹ Badiou attempts to demonstrate that T -sets defined over an external complete Heyting algebra give rise to a so called *Grothendieck-topos*—a topos of sheaves of sets over a category. As T is a set, it can be made a required category by deciding its elements to be its objects and the order relations between its elements the morphisms. He introduces the following notation.

5.1. Definition. The self-identity or existence in a T -set (A, \mathbf{Id}) is

$$\mathbf{E}x = \mathbf{Id}(x, x).$$

5.2. **Proposition.** *From the symmetry and transitivity of \mathbf{Id} it follows that⁴⁰*

$$\mathbf{Id}(x, y) \leq \mathbf{E}x \wedge \mathbf{E}y.$$

5.3. **Theorem.** *An object (A, \mathbf{Id}) whose every atom is real, is a sheaf.*

The demonstration that every object (A, \mathbf{Id}) is indeed a sheaf requires the definition of three operators: compatibility, order and localisation.

5.4. **Definition** (Localisation). For an atom $a : A \rightarrow T$, a *localisation* $a \int p$ on $p \in T$ is the atom which for each y establishes

$$(a \int p)(y) = a(y) \wedge p.$$

The fact that the resulting $a \int p : A \rightarrow T$ is an atom follows trivially from the fact that (the pull-back operator) \wedge is compatible with the order-relation. Because of the 'postulate of materialism', this localised atom itself is represented by some element $x_p \in A$. Therefore, the localization $x \int p$ also makes sense: $\mathbf{Id}(x \int p, y) = \mathbf{Id}(x, y) \wedge p$.

Two functions f and g may be *compatible* if they agree for every element of their common domain $f(x) = g(x)$. But when one only defines f and g as morphisms on objects (say X and Y) there needs to be a concept for determining the *localisations* $f|_{X \wedge Y} = g|_{X \wedge Y}$.

5.5. **Definition** (Compatibility). In Badiou's formalism, two atoms are *compatible* if

$$a \ddagger b \iff a \int \mathbf{E}b = b \int \mathbf{E}a.$$

5.6. **Proposition.** *I already declared that $\mathbf{Id}(a, b) \leq \mathbf{E}a \wedge \mathbf{E}b$; it is an easy consequence of the compatibility condition that if $a \ddagger b$, then $\mathbf{E}a \wedge \mathbf{E}b \leq \mathbf{Id}(a, b)$ and thus an equality between the two. This equality can be taken as a definition of compatibility.*

Sketch. The other implication entailed by the proposed, alternative definition has a bit lengthier proof. As a sketch, it needs first to be shown that $a \int \mathbf{Id}(a, b) = b \int \mathbf{Id}(a, b)$, and that the localization is transitive in the sense that $(a \int p) \int q = a \int (p \wedge q)$ ⁴¹, that is, it is compatible with the order structure. Such compatibility is obviously required in general sheaf theory, but unlike in the restricted case of locales, Grothendieck-topoi relativise (by the notion of a sieve) the substantive assumption of the order-relation. Only in the case of *locales* such

hierarchical sieve-structures not only appear as local, synthetic effects but are predetermined globally—ontologically by the poset-structure T . Finally, once demonstrated that $a \int (\mathbf{E}a \wedge \mathbf{E}b) = a \int \mathbf{E}b$, the fact that $a \ddagger b$ is an easy consequence⁴². \square

5.7. Definition (Order-Relation). Formally one denotes $a \leq b$ if and only if $\mathbf{E}a = \mathbf{Id}(a, b)$. This relation occurs now on the level of the object A instead of the Heyting algebra T . It is again an easy demonstration that $a \leq b$ is equivalent to the condition that both $a \ddagger b$ and $\mathbf{E}a \leq \mathbf{E}b$. Furthermore, it is rather straightforward to show that the relation \leq is reflexive, transitive, and anti-symmetric⁴³.

Proof of the Theorem 5.3. The proof of the sheaf-condition is now based on the equivalence of the following three conditions:

$$\begin{aligned} a &= b \int \mathbf{E}a && \iff \\ a \ddagger b \text{ and } \mathbf{E}a &\leq \mathbf{E}b && \iff \\ \mathbf{E}a &= \mathbf{Id}(a, b). \end{aligned}$$

These may be established by showing that $\mathbf{E}a = \mathbf{Id}(a, b)$, if and only if $a = b \int \mathbf{E}a$. The sufficiency of the latter condition amounts to first showing that $\mathbf{E}(a \int p) = \mathbf{E}a \wedge p$ ⁴⁴.

To proceed with the proof, we need to connect the previous relations to the envelope Σ . First, it needs to be shown that if $b \ddagger b'$, then

$$b(x) \wedge b'(y) \leq \mathbf{Id}(x, y)$$

for all x, y . This follows easily from the previous discussion. The crucial part is now to show that the function

$$\pi(x) = \Sigma\{\mathbf{Id}(b, x) \mid b \in B\}$$

is an atom if the elements of B are compatible in pairs⁴⁵. This is because it then retains a 'real' element which materialises such an atom. The first axiom (A1)

$$\mathbf{Id}(x, y) \wedge \pi(x) \leq \pi(y)$$

is straightforward⁴⁶. Now $\pi(x) \wedge \pi(y) = \Sigma\{\mathbf{Id}(b, x) \wedge \mathbf{Id}(b', y)\}$ and by the previous $\mathbf{Id}(b, x) \wedge \mathbf{Id}(b', y) \leq \mathbf{Id}(x, y)$ so $\mathbf{Id}(x, y)$ is an upper boundary, but since the previous Σ is the least upper bound, we have $\pi(x) \wedge \pi(y) \leq \mathbf{Id}(x, y)$. Therefore π is an atom and can denote by ϵ the the corresponding real element. Then it is possible to demonstrate

that $\mathbf{E}\epsilon = \Sigma\{\mathbf{E}b \mid b \in B\}$. It follows that ϵ itself is actually the least upper bound of B : there exists a real synthesis of B ⁴⁷.

Badiou characterises this 'transcendental functor of the object'⁴⁸, in other words this *sheaf*, as 'not exactly a function' as it associates rather than elements, 'subsets'. A sheaf can thus be expressed as a strata in which each neighborhood (transcendental degree) U becomes associated with the *set* of sections defined over U , usually denoted by $\mathcal{F}(U)$. Therefore, a sheaf is actually a *functor* $\mathcal{C}^{op} \rightarrow \mathcal{S}ets$, where \mathcal{C} is a category. In Badiou's restricted case \mathcal{C} is determined to be the particular kind of category \mathcal{C}_T ⁴⁹ deriving directly from the poset T . It results with a functor $\mathcal{C}_T^{op} \rightarrow \mathcal{S}ets$ in the following manner. Formally, for an object A , define $\mathcal{F}_A(p) = \{x \mid x \in A \text{ and } \mathbf{E}x = p\}$. If there is any $y \in \mathcal{F}_A(p)$ with $\mathbf{E}y = p$, then the equation $\mathbf{E}(y \int q) = \mathbf{E}y \wedge q$ amounts to $\mathbf{E}(y \int q) = p \wedge q$. If $q \leq p$ then $\mathbf{E}y \int q = q$ giving rise to a commutative diagram:

$$\begin{array}{ccc} p & \xrightarrow{\mathcal{F}_A} & \mathcal{F}_A(p) \\ \downarrow \leq & & \downarrow \int q \\ q & \xrightarrow{\mathcal{F}_A} & \mathcal{F}_A(q) \end{array}$$

which guarantees \mathcal{F}_A to be a functor and thus a presheaf.

Finally, one needs to demonstrate the sheaf-condition, the 'real synthesis' in Badiou's terminology. The functor $J(p) = \{\Theta \mid \Sigma\Theta = p\}$ forms a 'basis' of a so called Grothendieck-topology on T (see the next section). Let me now consider such a basis Θ and a collection x_q of elements where $q \in \Theta \in J(p)$. it derives from an imaginary section x_p , the elements would be pairwise compatible. In such a case, let me assume that they satisfy the 'matching' condition $x_q \int (q \wedge q') = x_{q'} \int (q \wedge q')$, which implies that $x_q \ddagger x_{q'}$. One would thus like to find an element x_p , where $x_q = x_p \int q$ for all $q \in \Theta$. But to demonstrate that the elements x_q commute in the diagram, they need to be shown to be pairwise compatible. One thus chooses x_p to be the envelope $\Sigma\{x_q \mid q \in \Theta\}$ and it clearly satisfies the condition. Namely, we just demonstrated that the envelope $\Sigma\{x_q\}$ localises to x_q for all q , that is,

$$\Sigma\{x_q\} \int q = x_q, \quad \forall q,$$

and that $\mathbf{E}\Sigma\{x_q\} = p$. The fact that it is unique then will do the trick. In the vocabulary of the next section, we have thus sketched Badiou's

proof that objects are those of the topos $\mathcal{S}hvs(T, J)$ (see the following remark). \square

As a result, Badiou nearly succeeds in establishing the first part of his perverted project to show that T -sets form a topos (while claiming to work on topos theory more broadly). In other words, T -sets are 'capable of lending consistency to the multiple' and express sheaf as a set 'in the space of its appearing'⁵⁰.

5.8. *Remark.* As a final remark, what Badiou disregards in respect to the definition of an object, given a region $B \subset A$, whose elements are compatible in pairs, he demonstrates that the function $\pi(x) = \Sigma\{\mathbf{Id}(b, x) \mid b \in B\}$ is an atom of A . If one wants to consider the smallest \bar{B} containing B within A which itself is an object, the 'real' element representing $\pi(x)$ should by the postulate of materialism itself lie in the atom: $\epsilon \in \bar{B}$ and thus because all elements are compatible, every element $b' \in \bar{B}$ has $b' \leq b$. Therefore, the sub-objects of the object A are generated by the ideals $\downarrow(\epsilon)$, each of them purely determined by the arrow $1 \rightarrow \{\epsilon\} \rightarrow \bar{B}$. (See remark 6.8.)

6. CATEGORICAL AND CHANGING FORMS OF 'MATERIALITY'

As discussed in the previous section, category theory stands for the broader tendency in which mathematics moves away from the questions of contents and consistence of being towards those related to one's composition, coherence and *being-there*: the question of 'taking place' in a topos. *Topos* is a Greek phrase that refers to a 'place' or a 'commonplace', indeed, and it quite interestingly relates phenomenological concepts like 'taking place' and 'being-there' to mathematics. This has to do with Kant's reasoning but also the long discussion that followed. It particularly contests Badiou's own 'Platonic' position on the so called 'calculated phenomenology', whose formal foundations (the theory of external complete Heyting algebras) was expressed above.

Instead of denouncing the question of being, however, we would rather say that topos theory makes the two discourses *intersect*: the ontological one of being and the phenomenological one of being-there. Sheaf theory was the first implication of this encounter, and it is crucial to understand sheaf theory both to understand Badiou's own aims but also to introduce the definition of Grothendieck-topoi which also

inspired the broader definition of elementary topoi. A sheaf in particular is a functor which composes different sections so as to categorically 'represent' the idea of a *function* without referring to it on the level of set-theoretic elements.

6.1. **Definition.** If X is a topological space, then for each open set U the sheaf \mathcal{F} consists of an object $\mathcal{F}(U) \in \text{Ob}(\mathcal{C})$ satisfying two conditions. First, there needs to exist certain 'natural' inclusions, so called restrictions

$$\mathcal{F}(U) \rightarrow \mathcal{F}(U') : f_U \mapsto f_U|_{U'}$$

for all open sets $U' \subset U$. There is also a *compatibility condition* which states that given a collection $f_i \in \mathcal{F}(U_i)$, there needs to be $f \in \mathcal{F}(\bigcup_i U_i)$ so that $f|_{U_i} = f_i$ for all i . In the categorical language, the sheaf-condition means that the diagram

$$\mathcal{F}(U) \rightarrow \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_U U_j)$$

is a coequaliser for each covering sieve (U_i) of U , usually determined as the set $J(U)$ of such sieves.

6.2. *Remark* (Representable sheaves). Inspired by sheaf theory, we need to consider them as the so called 'natural transformations' (see 6.2, p. 18). Such transformations are dealt with by the so called Yoneda Lemma that is the starting point of any intelligible application of category theory.

Indeed, based on Yoneda lemma, the transformation from 'classical spaces' to sheaves is natural in the sense that given any 'classical space'

denoted as an object A , the natural transformations $\mathcal{C} \begin{array}{c} \xrightarrow{h_A} \\ \Downarrow \\ \xrightarrow{\mathcal{F}} \end{array} \mathcal{C}$

between the functors h_A and \mathcal{F} corresponds to the elements of $\mathcal{F}(A)$. If 'spaces' are regarded themselves equated with their corresponding (pre)sheaves h_A , we then have $F(A) = \text{Hom}(A, F)$. This gives as the famous Yoneda embedding

$$\mathbf{y} : \mathcal{C} \rightarrow \mathcal{S}ets^{\mathcal{C}^{op}},$$

which embeds the index-category \mathcal{C} into its corresponding Grothendieck-topos $\mathcal{S}ets^{\mathcal{C}^{op}}$. It maps objects of \mathcal{C} into *representable sheaves* in $\mathcal{S}ets^{\mathcal{C}^{op}}$.

6.3. *Remark.* The above definition is based on the Bourbakian, set-theoretic definition of topology as a collection of open subsets (X) (which is a locale). It gives rise to a similar structure Badiou defines as the 'transcendental functor', that is, a sheaf identified with the set-theoretically explicated *functional strata*. Topos theory makes an alternative, categorical definition of topology—for example the so called Grothendieck-topology. Regardless of the topological framework, the idea is to categorically impose such structures of localization on *Sets* that enforce 'points', and thus the 'space' consisting of such points, to emerge. Whereas set theory presumes the initially atomic primacy of points, the sheaf-description of space does not assume the result of such localization—the actual points—as analytically given but only synthetic result of the functorial procedure of localisation. However, inasmuch as sheaf-theory—and topos theory—aims to bridge the categorical register with the 'ontological' sphere of sets, such a strata of sets is still involved even in the case of topos theory. For a more general class of such categorical stratifications, one may move to the theory of grupoids and stacks, that is, categories fibred on grupoids.

6.4. **Definition** (Grothendieck-topology). Let \mathcal{C} be a category. A *sieve*—in French a 'crible'—on C is a covering family of C so that it is downwards closed. A *Grothendieck-topology* on a category \mathcal{C} is a function J assigning a collection $J(C)$ of sieves for every $C \in \mathcal{C}$ such that

- 1) the maximal sieve $\{f \mid \text{any } f : D \rightarrow C\} \in J(\mathcal{C})$,
- 2) (stability) if $S \in J(C)$ then $h^*(S) \in J(D)$ for any $h : D \rightarrow C$, and
- 3) (transitivity) if $S \in J(C)$ and R is any sieve on C such that $h^*(R) \in J(D)$ for all arrows $h : D \rightarrow C$, then $R \in J(C)$.

The stability condition specifies that the intersections of two sieves is also a sieve. Given the latter two conditions, a topology itself is a sheaf on the maximal topology consisting of all sieves. It is often useful to consider a basis; for example Badiou works on such a 'basis' in the previous proof without explicitly stating this.

6.5. **Definition.** A basis K of a Grothendieck-topology J consists of collections (not necessarily sieves) of arrows:

- 1) for any isomorphism $f : C' \rightarrow C$, $f \in K(C)$,

- 2) if $\{f_i : C_i \rightarrow C \mid i \in I\} \in K(C)$, then the pull-backs along any arrow $g : D \rightarrow C$ are contained in $\{\pi_2 : C_i \times_C D \rightarrow D \mid i \in I\} \in K(D)$, and
- 3) if $\{f_i : C_i \rightarrow C \mid i \in I\} \in K(C)$, and if for each $i \in I$ there is a family $\{g_{ij} : D_{ij} \rightarrow C_i \mid j \in I_i\} \in K(C_i)$, also the family of composites $\{f_i \circ g_{ij} : D_{ij} \rightarrow C \mid i \in I, j \in I_i\}$ lies in $K(C)$.

For such a basis K one may define a sieve $J_K(C) = \{S \mid S \supset R \in K(C)\}$ that generates a topology.

6.6. *Remark.* The notion of Grothendieck-topology enables one to generalise the notion of a sheaf to a broader class of categories instead of the (classical) category of Kuratowski-spaces. If \mathbf{esp} is the category of topological spaces and continuous maps, sheaves on $X \in \text{Ob}(\mathcal{C})$ are actually equivalent to the full subcategory of $\mathbf{esp}/(X, \mathcal{C})$ of local homeomorphism $p : E \rightarrow X$. The corresponding pair of adjoint functors $\mathcal{S}ets^{\mathcal{C}^{op}} \rightleftarrows \mathbf{esp}/(X, \mathcal{C})$ yields an associated sheaf functor $\mathcal{S}ets^{\mathcal{C}^{op}} \rightarrow \mathcal{S}hvs(X)$ which shows that sheaves can be regarded either as presheaves with exactness condition or as spaces with local homeomorphisms into X ⁵¹.

6.7. *Remark.* In the case of Grothendieck-topology, there is an equivalence of categories between the category of pre-sheaves $\mathcal{S}ets^{\mathcal{C}^{op}}$ and the category of sheaves $\mathcal{S}hvs(\mathcal{C}, J)$ where J is the so called canonical Grothendieck-topology. Thus one often omits the reference to particular topology and deals with presheaves $\mathcal{S}ets^{\mathcal{C}^{op}}$ instead, even if their objects do not satisfying the two sheaf-conditions.

6.8. *Remark.* Now we have introduced the formalism required to accomplish the final step of Badiou's proof in the previous section (see remark 5.8). Namely, the category $\mathcal{S}ets^{T^{op}}$ is generated by representable presheaves of the form $y(p) : q \mapsto \text{Hom}_T(q, p)$ by the Yoneda lemma. Let

$$a : \mathcal{S}ets^{T^{op}} \rightarrow \mathcal{S}hvs(T, J)$$

be the associated sheaf-functor $P \mapsto P^{++}$ ⁵². Then because T is a poset, for any $p \in T$, the map $\text{Hom}(\cdot, p) \rightarrow 1$ is a mono, and because a is left-exact, also $ay(p) \rightarrow 1$ is a mono⁵³. Through it, any sheaf is a subobject of 1. This is exactly the axiom of support generators (SG) that is crucial to Badiou's constitutive postulate of materialism: it follows exactly from the organization of T as an ordered poset and

this order-relation being functorially extendible to general objects of $\mathcal{S}hvs(T, J)$.

7. HISTORICAL INSIGHT: FROM CATEGORIES TO GEOMETRY

But what makes categories historically remarkable and, in particular, what demonstrates that the categorical change is genuine? Of course, as we have done, it is one thing to discuss and demonstrate how Badiou fails to show that category theory is not genuine. But it is another thing to say that mathematics itself does change, and that the 'Platonic' *a priori* in Badiou's endeavour is insufficient. For this we need something more tangible, something more *empirical*.

Yet the empirical does not need to stand only in a way opposed to mathematics. Rather, it relates to results that stemmed from and would have been impossible to comprehend without the use of categories. It is only through experience that we are taught the meaning and use of categories. An experience obviously absent from Badiou's habituation in mathematics.

To contrast, Grothendieck opened up a new regime of algebraic geometry by generalising the notion of a space first scheme-theoretically (with sheaves) and then in terms of grupoids and higher categories⁵⁴. Topos theory became synonymous to the study of categories that would satisfy the so called Giraud's axioms⁵⁵ based on Grothendieck's geometric machinery (though only a limited part of it). By utilising such tools, Pierre Deligne was able to prove the so called Weil conjectures, mod- p analogues of the famous Riemann hypothesis.

These conjectures—anticipated already by Gauss—concern the so called local ζ -functions that derive from counting the number of points of an algebraic variety over a finite field, an algebraic structure similar to that of for example rational \mathbb{Q} or real numbers \mathbb{R} but with only a finite number of elements. By representing algebraic varieties in polynomial terms, it is possible to analyse geometric structures analogous to Riemann hypothesis but over finite fields $\mathbb{Z}/p\mathbb{Z}$ (the whole numbers modulo p). Such 'discrete' varieties had previously been excluded from topological and geometric inquiry, while it now occurred that geometry was no longer overshadowed by a need to decide between 'discrete' and 'continuous' modalities of the subject (that Badiou still separates).

Along with the continuous ones, also discrete variates could then be studied based on Betti numbers, and similarly as what Cohen's argument made manifest in set-theory, there seemed to occur 'deeper', topological *precursors* that had remained invisible under the classical formalism. In particular, the so called étale-cohomology allowed topological concepts (e.g., neighbourhood) to be studied in the context of algebraic geometry whose classical, Zariski-description was too rigid to allow a meaningful interpretation. Introducing such concepts on the basis of Jean-Pierre Serre's suggestion, Alexander Grothendieck did revolutionarize the field of geometry, and Pierre Deligne's proof of the Weil-conjectures, not to mention Wiles' work on Fermat's last theorem that subsequently followed.

Grothendieck's crucial insight drew on his observation that if morphisms of varieties were considered by their 'adjoint' field of functions, it was possible to consider geometric morphisms as equivalent to algebraic ones. The algebraic category was restrictive, however, because field-morphisms are always monomorphisms which makes geometric morphisms epis: to generalize the notion of a neighbourhood to algebraic category he needed to embed algebraic fields into a larger category of rings. While a traditional Kuratowski covering space is locally 'split'—as mathematicians call it—the same was not true for the dual category of fields. In other words, the category of fields did not have an operator analogous to pull-backs (fibre products) unless considered as being embedded within rings from which pull-backs have a co-dual expressed by the tensor operator \otimes . Grothendieck thus realized he could replace 'incorporeal' or contained neighborhoods $U \hookrightarrow X$ by a more relational description: as maps $U \rightarrow X$ that are not necessarily monic, but which correspond to ring-morphisms instead.

Topos theory applies similar insight but not in the context of only specific varieties but for the entire theory of sets instead. Ultimately, Lawvere and Tierney realized the importance of these ideas to the concept of classification and truth in general. Classification of elements between two sets comes down to a question: does this element belong to a given set or not? In category of *Sets* this question calls for a binary answer: true or false. But not in a general topos in which the composition of the subobject-classifier is more geometric.

Indeed, Lawvere and Tierney then considered this *characteristic* map 'either/or' as a categorical *relationship* instead without referring to its 'contents'. It was the structural form of this morphism (which they called 'true') and as contrasted with other relationships that marked the beginning of *geometric logic*. They thus rephrased the binary complete Heyting algebra of classical truth with the categorical version Ω defined as an object, which satisfies a specific pull-back condition. The crux of topos theory was then the so called Freyd–Mitchell embedding theorem which effectively guaranteed the explicit set of elementary axioms so as to formalize topos theory.

We will come back to this definition later. But to understand its significance as a link between geometry and language, it is useful to see how the characteristic map (either/or) behaves in set theory. In particular, by expressing truth in this way, it became possible to reduce Axiom of Comprehension⁵⁶, which states that any suitable formal condition λ gives rise to a peculiar set $\{x \in \lambda\}$, to a rather elementary statement regarding adjoint functors⁵⁷.

At the same time, many mathematical structures became expressible not only as general topoi but in terms of a more specific class of Grothendieck-topoi. There, too, the 'way of doing mathematics' is different in the sense that the object-classifier is categorically defined and there is no empty set (initial object) but mathematics starts from the terminal object 1 instead. However, there is a material way to express the 'difference' such topoi make in terms of set theory: for every such a topos there is a sheaf-form enabling it to be expressed as a category of sheaves $\mathcal{S}ets^{\mathcal{C}}$ for a category \mathcal{C} with a specific Grothendieck-topology.

8. POSTULATE OF MATERIALISM—OR TWO POSTULATES INSTEAD?

Let me now discuss the postulate of materialism from the categorical point of view to see how and why geometry matters to the question of materiality, and even in relation to set theory. Badiou's own definition is a strong or constrained one and we will explicate the condition which enables us to specify whether a 'weakly material' topos is actually local, that is, constitutively material in Badiou's strong sense of

the term. Therefore, we begin with a more general, abstract expression of Badiou's objects (in terms of elementary theory) even if they do satisfy also the more specific conditions (local theory).

Indeed, Badiou's objects may always be interpreted as Grothendieck-topoi (situated in $\mathcal{S}ets^{Top}$) which is a 2-category distinct from the more abstractly designated 2-category elementary topoi ($\mathcal{T}op$). An elementary topos \mathcal{E} is a Grothendieck-topos only if there is a particular morphism $\mathcal{E} \rightarrow \mathcal{S}ets$ that 'materialises' its internal experience of truth even if this morphism wasn't 'bounded' in the sense that the internal logic of the topos \mathcal{E} isn't reducible to the external logic of $T\text{-}\mathcal{S}ets$ (or $\mathcal{S}ets$).

8.1. Postulate (Weak Postulate of Materialism). Categorically the weaker version of the postulate of materialism signifies precisely condition which makes an elementary topos a so called Grothendieck-topos (see remark 10.3). In particular, it is a topos \mathcal{E} with a specific *materialization* morphism $\mathcal{E} \rightarrow \mathcal{S}ets$.

Let us now discuss how this weak postulate of materialism relates to what Badiou takes as the 'postulate of materialism'. A Grothendieck-topos can generally be written as $\mathcal{S}ets^{\mathcal{C}^{op}}$ defined over any category \mathcal{C} in which points might have internal automorphisms for example. This means that the 'sections' do *materialise* in the category of $\mathcal{S}ets$ but its *internal structure* is still irreducible to this material, set-theoretic surface of appearance. In Badiou's case it is this this internal structure which is not only material but *bounded* in its very materiality. In effect, if the underlying 'index-category' $\mathcal{C} = \mathcal{C}_T$ a *poset* corresponding to the external Heyting algebra T , any internal 'torsion' such as non-trivial endomorphisms of \mathcal{C} are extensively forced out.

What is crucial to Badiou's demonstration is that objects are atomic—that every atom is 'real'. That is what we phrase as the *postulate of atomism* and in elementary topos theory it is often phrased as the axiom regarding the support of generators (SG). Let us overview its categorical meaning. It accounts for the unity and singularity of the terminal object 1. In general, an object may be said to retain 'global elements' $1 \rightarrow X$, but there are also 'local elements' $U \rightarrow X$ that might not be determinable by global elements alone. For each arrow $U \rightarrow X$, a 'local' element on U , the pull-back object U^*X in the local topos \mathcal{E}/U has exactly one global element of U^*X , that is, a local element of X

over U ⁵⁸. The object X , as a whole, cannot be determined by 'global elements' alone, if there occurs torsion that obstructs such a dominant hierarchy pertinent to the category of $T\text{-}\mathcal{S}ets$ —a poset-structure analogous to Cohen's forcing-structure and that Badiou associated with his '*fundamental law of the subject*'⁵⁹. Only if such a hierarchy is given once and for all, does geometric intuition become superfluous. But for example in the case of the Möbius strip, that is, the non-trivial $\mathbb{Z}/2\mathbb{Z}$ -bundles give rise to a category which cannot be similarly forced into a hierarchical one.

On the basis of the extent to which elements in a X are discerned by its global elements, one can now phrase the postulate of atomism in the categorical framework. Of course, unless working in local theory, this does not yet result with the complete 'postulate of materialism' in Badiou's sense. Rather, we need to combine that with the weak postulate of materialism.

8.2. Axiom (Postulate of Atomism). An elementary topos \mathcal{E} *supports generators* (SG) if the subobjects of 1 in the topos \mathcal{E} generate \mathcal{E} . This means that given any pair of arrows $f \neq g : X \rightrightarrows Y$, there is $U \hookrightarrow X$ such that the two induced maps $U \rightrightarrows Y$ do not agree. In the case of \mathcal{E} being defined over the topos of $\mathcal{S}ets$, the axiom (SG) is equivalent to the condition, that the object 1 alone generates the topos \mathcal{E} ⁶⁰. If \mathcal{E} satisfies (SG), then Ω is a cogenerator of \mathcal{E} —differentiating between a parallel pair from the right—and for any topology j or an internal poset \mathbf{P} , both $\mathcal{E}^{\mathbf{P}}$ and $\mathcal{S}hvs_j(\mathcal{E})$ satisfy (SG)⁶¹.

Based on the two postulates—the weak postulate of materialism and the postulate of atomism—it is now possible to draw the general and abstract *topos-theoretic conditions* of Badiou's 'postulate of materialism' which should rather be phrased as the *postulate of atomic materialism*.

8.3. Postulate (Strong Postulate of Materialism). In addition to the weak postulate of materialism, the *strong postulate of materialism* presumes that a Grothendieck-topos defined over $\mathcal{S}ets$ should additionally satisfy the postulate of atomism, that is, that its objects are generated by the subobjects of the terminal object 1. This is the postulate of atomic materialism.

According to the following proposition, the topoi satisfying the strong postulate are shown to be *local topoi* (over $\mathcal{S}ets$) and *vice versa*. Namely,

for any such topos one chooses the transcendental $T = \gamma_*(\Omega)$ for its morphism of *materialist appearance* γ , and the equivalence then follows.

8.4. Proposition. *If an elementary topos \mathcal{E} is Grothendieck-topos defined over $\gamma : \mathcal{E} \rightarrow \mathcal{S}ets$, then it supports generators if and only if $\gamma : \mathcal{E} \rightarrow \mathcal{S}ets$ is logical and thus bounded. This means that the internal complete Heyting algebra Ω transforms into an external complete Heyting algebra $\gamma_*(\Omega)$ and then \mathcal{E} is equivalent to the topos $\mathcal{S}ets^{\gamma_*(\Omega)^{op}}$.*

8.5. Remark. In general topos theory, the subobjects (monics) of the internal subobject-classifier Ω form a class of arrows $\text{Hom}(\Omega, \Omega)$ as much as the global elements of Ω , that is, the arrows $1 \rightarrow \Omega$, corresponds to the subobjects of 1. For any object U there are also the subobjects $U^*\Omega$ corresponding to arrows $U \rightarrow \Omega$, so called *generalised points* whereas the localic theory of T -sets fails to resonate with such generalisation. Therefore, the designation of T as a set of global points is possible *only* for Grothendieck-topoi that support generators.

8.6. Remark. Obviously, if one works on an elementary topos \mathcal{E} which is not defined over $\mathcal{S}ets$ (contrary to the case of Grothendieck-topoi), nothing guarantees that the axiom SG would imply the topos to be a locale. The implication is valid only when \mathcal{E} is a Grothendieck-topos in the first place. The axiom SG could thus be taken as an alternative but incomplete version of the 'postulate of materialism', whereas the Grothendieck-condition together makes it strong or constitutive in Badiou's sense. But because SG alone does not guarantee the existence of the materialization morphism $\gamma : \mathcal{E} \rightarrow \mathcal{S}ets$, we can say that the strong postulate of materialism is SG *when* the weak condition already applies. In other words, it makes sense to treat the two versions of the postulate in a hierarchical fashion so that one is contained in the other.

9. MATHEMATICAL FRONTIER OF DIALECTICAL MATERIALISM?

We now focus on the weak postulate of materialism. What makes topoi satisfying this condition particularly prone to reflecting the structure of Badiou's own, 'material dialectic' reasoning (which he then falsely believed to be unmathematical)? In fact, it is possible to say that the two postulates do mathematically embody the distinction between 'dialectical' and 'democratic materialism': that a local topos

(where the strong postulate applies) is 'democratic' in the sense that there is only one dominant and thus 'general' will (T) materialized in sets. Or as Badiou says, it is a topos where 'existence = individual = body'.

This is, of course, very different from Deleuze's view of democracy even if Badiou particularly mentions Deleuze in this context. Indeed, as opposed to post-structuralism, in a local topos there is only one body of truth whereas a Grothendieck-topos articulates the incoherence of bodies (at least tentatively).

An atomic world thus prohibits subjective torsion as opposed to weakly materialist one where the 'fundamental law of the subject' constitutive to Badiou's view on mathematics no longer applies. Precisely by allowing torsion but regulating the object by categorical means, a Grothendieck-topos gives rise to an alternative notion of truth that should be taken as a *mathematical basis* for Badiou's 'material dialectic'.

Thus far we have postponed the formal definition of Grothendieck-topoi, even if such structures were considered while phrasing the weak postulate. There are two ways—set-theoretically material and categorically abstract—ways to introduce that concept. To begin with the first definition, *sheaves* are defined in relation to three operators: compatibility, order and localization. In the case of a traditional (ontological) topological space X , a sheaf is just a presheaf $\mathcal{F} \in \mathcal{S}ets^{(X)}$ satisfying compatibility condition. This definition of a sheaf applicable to a locale (X) has, however, a flaw. Rather than there being an externally given poset of localizations (eg. (X)) *once and for all*, we need to consider a multitude of such posets which are more or less compatible due to 'torsion'.

Grothendieck followed this insight while he worked on étale-theory: he didn't reject the possibility of torsion in general while he still managed to consider algebraic structure in a *material enough* way to relate them to set theory, that is, to allow them to materialize a more 'traditional' outlook on algebraic geometry which would be applicable at least locally in respect to a given hierarchy or domination. In other words, Grothendieck-topologies became to provide suitable 'snapshots' that were locally constrained, hierarchical—this allowed local sites to be dealt with logically without prohibiting torsion on the global scale.

This technique proved helpful in multiple settings. In general topos theory, the entire notion of topology is consequentially replaced whereas Grothendieck's work on étale-theory still requires a (set-theoretically) material, even if altered definition of topology. The latter in fact allows the 'immaterial', that is, the torsion of the subject to materialize in a way that is not only categorically abstract but consequential to Badiou's own, 'ontological' world of reason.

To formalize this trade-off between algebra and geometry, Grothendieck solved the problem of torsion—the treatment of the immaterial— by means that are locally hierarchical, that is, they are (only) locally compatible with Badiou's strong postulate. Each such a hierarchical correlate is a so called *sieve*, while the topology as a whole combines a synthesis over an entire class of them. Namely, a sieve is a structure that concerns coverings $U_i \rightarrow X$ (of non-monic, that is, non-injective neighbourhoods) but whose index-category is subject to 'partial' or 'local' hierarchies or filters—precisely the sieves. To contrast these structures with Badiou's reduced setting in which the question of *a priori* is still overshadowed by an 'analytic', *dominant* hierarchy or sieve, this indexing is *retroactive* or 'synthetic' and may combine multiple mutually incompatible hierarchies. It is external to the underlying 'ontological', set-theoretic structure.

9.1. Definition (Material definition). A *Grothendieck-topology* J on a category \mathcal{C} associates a collection $J(C)$ of *sieves on* C to every object $C \in \mathcal{C}$, that is, it is a downward closed covering families on C (see definition 6.4).

9.2. Remark. If $U \rightarrow X$ lies in $J(C)$, then any arrow $V \rightarrow U \rightarrow X$ has to reside there as well. Thus, the association law makes any sieve a *right ideal*. In general, if the functor $y : C \rightarrow \text{Hom}(\cdot, C) : \mathcal{C} \rightarrow \mathcal{S}ets$ is considered as a presheaf $y : \mathcal{C} \rightarrow \mathcal{S}ets^{\mathcal{C}^{op}}$, then a sieve is a *subobject* $S \subset y(C)$ in the category of presheaves $\mathcal{S}ets^{\mathcal{C}^{op}}$.

9.3. Definition (Closed sieve). A sieve S on C is called *closed* if and only if for all arrows $f : D \rightarrow C$, the pull-back $f^*M \in J(D)$ implies $f \in M$, which in turn implies that f^*M is the maximal sieve on D .

9.4. Remark. Similar structures are employed in the case of Cohen's procedure, and thus also in the *Being and Event*. The so called 'correct sets' $\wp \in \mathcal{C}$ are *closed* similarly to sieves: whenever $p \in \wp$ and $q \leq p$,

then $q \in \wp$. If such a 'multiple' \wp is 'generic'— that is, if it intersects every domination—it 'covers' the whole space.

9.5. Definition (Subobject classifier). For a Grothendieck-site $\mathcal{S}hvs(\mathcal{C}, J)$, we can be define a subobject-classifier as

$$\Omega(C) = \text{the set of closed sieves on } C,$$

which, in fact, is not only a presheaf but a sheaf since the condition of closedness in the category of presheaves, Ω could be defined to consist of all sieves. In the context of *Being and Event*, we can consider the subobject-classifier as set of 'generic' filters relative to S .

9.6. Remark. In fact, given a sieve S on C , one may define a closure of S as⁶²

$$\bar{S} = \{h \mid h \text{ has a codomain } C, \text{ and } S \text{ covers } h\}.$$

The closure operation is actually organic to the categorical definition of topology in elementary theory. In general, one defines topology in respect to the subobject-classifier Ω as an arrow $j : \Omega \rightarrow \Omega$ with $j^2 = j$, $j \circ \text{true} = \text{true}$ and $j \circ \wedge = \wedge \circ (j \times j)$, where the morphism true is a unique map $1 \rightarrow \Omega$ related to the definition of an elementary topos (see definition 10.2).

As we pointed out above, in the *Logics of Worlds* Badiou⁶³ himself defines a Grothendieck-topology when he demonstrates the existence of the 'transcendental functor'⁶⁴, that is, the sheaf-object on the basis of a transcendental grading T . Badiou⁶⁵ defines the 'territory of p ' as

$$K(p) = \{\Theta \mid \Theta \subset T \text{ and } p = \Sigma\Theta\}.$$

They form a basis of a Grothendieck-topology. In short, every territory gives a sieve if it is completed under \leq -relation of T .

9.7. Remark. To formalise the compatibility conditions that Badiou's proof requires, let me define these in terms of a Grothendieck-topology. A sieve S is called a *cover* of an object C if $S \in J(C)$. Let $P : \mathcal{C}^{op} \rightarrow \mathcal{S}ets$ be any presheaf (functor). Then a *matching family* for S of elements of P is a function which for each arrow $f : D \rightarrow C$ in S assigns an element $x_f \in P(D)$ such that $x_f \circ g = x_{fg}$ for all $g : E \rightarrow D$. The so called *amalgamation* of such a matching family is an element $x \in P(C)$ for which $x \circ f = x_f$. In Badiou's terminology, such matching

families are regarded as pairwise compatible subsets of an object. An amalgamation—a 'real' element incorporating an atom does not need to exist. P is defined as a *sheaf* when they do so uniquely for all $C \in \mathcal{C}$ and sieves $S \in J(C)$. In Badiou's example, a matching family $\{x_f \mid f \in \Theta \subset T\}$ corresponds to an atom

$$\pi(a) = \Sigma\{\mathbf{Id}(a, x_f) \mid f \in \Theta\},$$

and this is precisely what we established in the above proof of the sheaf-condition⁶⁶.

The topology with a basis formed by 'territories' on p is, in fact, subcanonical in the category of the elements of T where arrows are the order-relations. This is because for any $p \in T$ one has $\text{Hom}_T(q, p) = \{f_{qp} \mid q \leq p\}$, and therefore as a set $h_p = \{q \mid q \leq p\}$ becomes canonically indexed by T . This makes h_p bijective to a subset of T . Now for any territory Θ on p , a matching family of such subsets h_q are obviously restrictions of h_p . In general, for a presheaf P we can construct another presheaf which can then be completed as a sheaf, that is, it can be sheafified in order to give rise to the uniqueness condition of 'atoms' similar to that discussed in *Logics of Worlds*. Sheaves can always be replaced by presheaves and *vice versa* in practical calculations.

10. DIAGRAMMATIC, ABSTRACT TOPOS

As we have now discussed the definition of the more basic structure of Grothendieck-topoi, let us now focus on the more abstract, *elementary* definition of a topos and discuss (weak) *materiality* then in this categorical context. The materiality of being can, indeed, be defined in a way that makes no material reference to the category of *Sets* itself.

The stakes between being and materiality are thus reverted. From this point of view, a Grothendieck-topos is not one of sheaves over sets but, instead, it is a topos which is not defined based on a specific geometric morphism $\mathcal{E} \rightarrow \mathcal{S}ets$ —a materialization—but rather a one for which such a materialization exists only when the topos itself is already intervened by an explicitly given topos similar to *Sets*. Therefore, there is no need to start with set-theoretic structures like sieves or Badiou's 'generic' filters. At the same time, the strong postulate receives a categorical version:

10.1. **Definition** (Strong Postulate, Categorical Version). For a given materialization the situation \mathcal{E} is faithful to the *atomic* situation of truth ($\mathcal{S}ets^{\gamma^*(\Omega)^{op}}$) if the *materialization morphism* itself is *bounded* and thus *logical*.

In particular, this alternative definition suggests that *materiality itself is not inevitably a logical question*⁶⁷.

Therefore, for this definition to make sense, let us look at the question of materiality from a more abstract point of view: what are topoi or 'places' of reason that are not necessarily material or where the question of materiality differs from that defined against the 'Platonic' world of $\mathcal{S}ets$? Can we deploy the question of materiality without making any reference—direct or sheaf-theoretic—to the question of what the objects 'consist of', that is, can we think about materiality without crossing Kant's categorical limit of the object? Elementary theory suggests that we can.

10.2. **Definition** (Elementary Topos). An *elementary topos* \mathcal{E} is a category which

- 1) has finite limits, or equivalently \mathcal{E} has so called pull-backs and a terminal object 1 ⁶⁸,
- 2) is Cartesian closed, which means that for each object X there is an *exponential functor* $(-)^X : \mathcal{E} \rightarrow \mathcal{E}$ which is right adjoint to the functor $(-) \times X$ ⁶⁹, and finally
- 3) (axiom of truth) \mathcal{E} retains an object called the *subobject classifier* Ω , which is equipped with an arrow $1 \xrightarrow{\text{true}} \Omega$ such that for each monomorphism $\sigma : Y \hookrightarrow X$ in \mathcal{E} , there is a unique *classifying map* $\varphi_\sigma : X \rightarrow \Omega$ making $\sigma : Y \hookrightarrow X$ a pull-back⁷⁰ of φ_σ along the arrow *true*.

10.3. *Remark* (Grothendieck-topos). In respect to this categorical definition, a Grothendieck-topos is a topos with the following conditions (J. Giraud)⁷¹ satisfies: (1) \mathcal{E} has all set-indexed coproducts, and they are disjoint and universal, (2) equivalence relations in \mathcal{E} have universal coequalisers. (3) every equivalence relation in \mathcal{E} is effective, and every epimorphism in \mathcal{E} is a coequaliser, (4) \mathcal{E} has 'small hom-sets', i.e. for any two objects X, Y , the morphisms of \mathcal{E} from X to Y are parametrized by a set, and finally (5) \mathcal{E} has a set of generators (not necessarily monic in respect to 1 as in the case of locales). Together the five conditions can

be taken as an alternative definition of a Grothendieck-topos (compare to definition 6.4), and therefore also to the weak postulate.

We should still demonstrate that Badiou's world of T -sets is actually the *category* of sheaves $\mathcal{S}hvs(T, J)$ and that it will, consequentially, hold up to those conditions of a topos listed above. To shift to the categorical setting, one first needs to define a relation between objects. These relations, the so called 'natural transformations' we encountered in relation Yoneda lemma, should satisfy conditions Badiou regards as 'complex arrangements'.

10.4. Definition (Relation). A *relation* from the object (A, \mathbf{Id}_α) to the object (B, \mathbf{Id}_β) is a map $\rho : A \rightarrow B$ such that

$$\mathbf{E}_\beta \rho(a) = \mathbf{E}_\alpha a \quad \text{and} \quad \rho(a \dot{\smile} p) = \rho(a) \dot{\smile} p.$$

10.5. Proposition. *It is a rather easy consequence of these two pre-suppositions that it respects the order relation \leq one retains*

$$\mathbf{Id}_\alpha(a, b) \leq \mathbf{Id}_\beta(\rho(a), \rho(b))$$

and that if $a \dot{\ddagger} b$ are two compatible elements, then also $\rho(a) \dot{\ddagger} \rho(b)$ ⁷². Thus such a relation itself is compatible with the underlying T -structures.

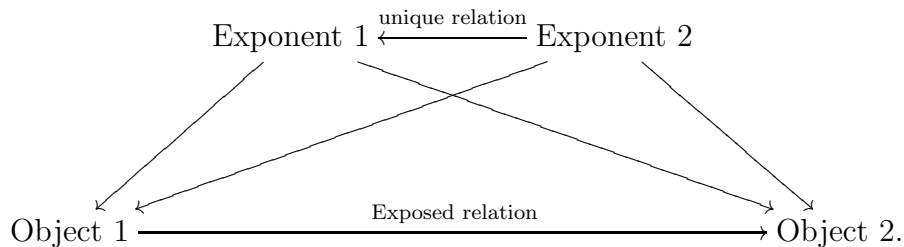
Given these definitions, regardless of Badiou's⁷³ confusion about the structure of the 'power-object', it is safe to assume that Badiou has demonstrated that there is at least a *category* of T - $\mathcal{S}ets$ if not yet a topos. Its objects are defined as T -sets situated in the 'world \mathbf{m} '⁷⁴ together with their respective equalization functions \mathbf{Id}_α . It is obviously Badiou's 'diagrammatic' aim to demonstrate that this category is a topos and, ultimately, to reduce any 'diagrammatic' claim of 'democratic materialism' to the constituted, non-diagrammatic objects such as T -sets. That is, by showing that the particular set of objects is a categorical makes him assume that every category should take a similar form: a classical mistake of reasoning referred to as *affirming the consequent*. In the next section, we discuss Badiou's struggles regarding this affair. This is not only a technical issue but philosophically alarming as well.

11. BADIOU'S DIAGRAMMATIC STRUGGLE

Confusing general topoi with specific locales, Badiou then believes to have overcome the need to work categorically, through 'diagrams', and

he again returns to his secure, 'Platonic' basis. Thus he believes to have reverted the course of topos theory by bringing categorical reasoning back down to set theory. By this he inevitably strikes the notion of 'diagrams' utilized not only by mathematicians but by several 'post-structuralists' after the term was first launched by Michel Foucault⁷⁵. However, he not only makes a converse error but also struggles in the way he treats his own, specific category. No question his erudition in categorical diagrammatics is deficient.

For example, when it comes to the actual 'mathemes' in this category of $T\text{-Sets}$, he demonstrates that there exists an object which he calls an 'exponent' of a relation $X \rightarrow Y$. In other words, '[i]f we consider [...] two Quebecois in turn as objects of the world, we see that they each entertain a relation to each of the objects linked together by the 'Oka incident' [...] and, by the same token, a relation to this link itself, that is a relation to a relation'⁷⁶. In diagrammatic terms, Badiou⁷⁷ defines a relation to be "universally exposed' if given two distinct expositions of the same relation, there exists between the two exponents [sic] one and only one relation such that the diagram remains commutative', and for that he draws the following diagram:



On the basis of this diagram, Badiou⁷⁸ argues that if the previous diagram with the 'progressive Qubebecois citizen' (Exponent 1) was similarly exposed also with another, 'progressive citizen' (Exponent 2), then 'if the reactionary supports the progressive, who in turn supports the Mohawks, there follows a flagrant contradiction with the direct relation of vituperation that the reactionary entertains with the Mohawks'. The claims is unwarranted: Badiou *falsely argues for the necessity of its non-commutativity*, even if it were possible, thus again confusing the necessary and the sufficient.

11.1. Proposition. *Badiou's condition of universal exposition is falsely stated. Given any relation $f : X \rightarrow Y$, it is easy to construct an*

object that does not satisfy Badiou's universality condition in almost any topos, in particular in $\mathcal{S}ets$.

Proof. To illustrate, let us assume that $T = T_{\text{false} < \text{true}}$ so that we actually work in the category of $\mathcal{S}ets$ and can treat its 'elements' point-wise. For example, we can consider the diagram with π_1, π_2 denoting the two projection maps of the Cartesian product $X \times X \rightarrow X$,

$$\begin{array}{ccc}
 X \times X \times X & \overset{h?}{\dashrightarrow} & X \times X \times X \\
 \downarrow \pi_1 & \nearrow \pi_2 & \downarrow f \circ \pi_2 \\
 X & \xrightarrow{f} & Y
 \end{array}$$

It is clear that the third component of the map h —namely the map $\pi_3 \circ h \circ \iota_3$ —can be defined arbitrarily because in any of the maps with the domain $X \times X \times X$ there are no references to the third projection-component. The only condition concerns the first component of h : it has to satisfy specific conditions with respect to the second component in the image of the map h . In other words, in Badiou's definition a relation is never universally exposed as long as there is enough room to play such as in the case of the category of $\mathcal{S}ets$.

There is further terminological confusion. This regards to how Badiou assumes the object to 'expose' a *single* relation. As such his 'exponent' obviously contradicts with the standard definition of category theory, considering it as the object X^Y of *all* relations between X and Y instead of exposing only a particular relation, say $f : X \rightarrow Y$. \square

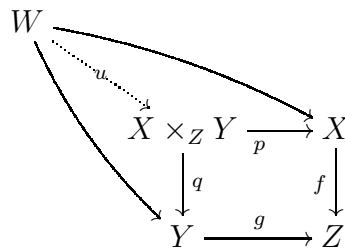
In short, Badiou's exponents are *graphs* in conventional terminology. The correct form of exposition—should it follow Badiou's reasoning in the case of the Quebecois—would then be that a relation is universally exposed by a *particular object* Γ_f so that whenever another object Z also exposes the relation, there is a unique arrow $Z \rightarrow \Gamma_f$, which in turn makes the diagram to commute. What is even more severe, however, is that Badiou forgets to specify that discussed object Γ_f (F_ρ in Badiou's example) *up to an isomorphism*—to specify what makes it structurally *unique*. It is this unique isomorphism class which should satisfy a so called *universal property* to specify it. The universal exposition of the relation f is not just a property of the relation alone but a particular

'universal object' in respect to that relation f ; in the above example it defines not any possible citizen but a unique form of a 'universal citizen' associated with the particular relation of the 'Oka incident'—call it $\Gamma_{\text{Oka incident}}$.

11.2. *Remark.* This inability to define what precisely is 'universal' in the object specified by the universal condition demonstrates that Badiou's understanding of what is specific to category theory is faulty. In other words, he fails to follow that precise 'Kantian' shift that doesn't specify objects directly, as what they *are*, but diagrammatically, in regard to how they *relate*. Such 'universal properties' are typical to modern mathematics in various branches, and they were used even before category theory was first introduced but as its structural precursors instead. Such universal properties then become the key to mathematics when dealt with from the categorical point of view. Badiou by contrast could not but denounce the diagrammatic approach in advance, even if he then falsely concluded to have 'shown' that the 'same mathematics' were still going on.

For example, the first condition in the general definition of an elementary topos \mathcal{E} requires there to exist finite limits⁷⁹ but this is equivalent to there existing a so called terminal object and pull-backs:

11.3. **Definition** (Pull-back). For any given two arrows $f : X \rightarrow Z$ and $g : Y \rightarrow Z$, a pull-back of those is an object $X \times_Z Y$ such that the inner diagram commutes if it satisfies the following universal condition: if the following diagram commutes, every arrow $u : W \rightarrow X \times_Z Y$ is unique up to an isomorphism.



11.4. **Proposition** (Exposition of a Singler Relation). *The corrected version of Badiou's universal 'exposition' of a particular relation now*

comes back to saying that there exists a pull-back

$$(1) \quad \begin{array}{ccc} F_\rho = \Gamma_{\rho:A \rightarrow B} & \xrightarrow{f} & X \\ \downarrow g & & \downarrow \rho \\ B & \xlongequal{1_B} & B \end{array}$$

in the category of T - \mathcal{S} ets.

11.5. *Remark.* With only a few more complications Badiou' proof of above proposition could easily be achieved by showing that *any* pull-back does exist. This would, in fact, complete the proof of the first condition of an elementary topos (definition 10.2). This may follow steps similar to the demonstration of the existence of a graph: something Badiou does manage to accomplish.

Partial proof. Badiou phrases the universal condition mistakenly and doesn't account to its *uniqueness*. He demonstrates only the *existence* of such a graph whereas uniqueness would need a corrected definition of a universal property. The proof of existence follows by first demonstrating that normal Cartesian product exists: '[g]iven two multiples A and B appearing in a world, the product of these two sets, that is the set constituted by all the ordered pairs of elements of A and B (in this order), must also appear in this world'⁸⁰. The next step consists of showing for a relation $\rho : (A, \mathbf{Id}_\alpha) \rightarrow (B, \mathbf{Id}_\beta)$ that the multiple consisting of pairs $(x, \rho(x)) \subset A \times B$, denoted by F_ρ is a multiple itself⁸¹, and if equipped with a map $\nu : F_\rho \rightarrow T$, where

$$\nu((a, \rho(a)), (b, \rho(b))) = \mathbf{Id}_\alpha(a, b) \wedge (\rho(a), \rho(b)),$$

this map satisfies the conditions of a transcendental indexing⁸²

$$\nu(x, y) \wedge \nu(y, z) \leq \nu(x, z).$$

To show that F_ρ is an object, one is thus required to show that '[e]very atom is real'⁸³. Given an atom $\epsilon : F_\rho \rightarrow T$ Badiou⁸⁴ constructs a map $\epsilon^* : a \mapsto \epsilon(a, \rho(a)) : A \rightarrow T$, which is an atom since ϵ satisfies the corresponding conditions. Hence by the 'postulate of materialism' it is real; say $\epsilon^*(x) = \mathbf{Id}_\alpha(c, x)$. But now $\epsilon(x, \rho(x)) = \nu[(c, \rho(c)), (x, \rho(x))]$ which proves Badiou's Lemma 4. By his Lemma 5, Badiou then demonstrates that the diagram (1) is valid: '[t]he object (F_ρ, ν) is an exponent of the

relation ρ' , which follows by showing that ' f and g conserve localizations', ie. $f(a \int p) = f(a) \int p$ and $g(a \int p) = g(a) \int p$; and that $(a, \rho(a)) \int p = a \int p$. Because of the definition of \mathcal{F}_p it is easy to see (using set theory) that it satisfies the universality condition of the pull-back diagram (1, p. 36) and thus Badiou⁸⁵ is left to show only that it similarly 'conserves existences and localizations'.

□

12. T -SETS FORM A TOPOS—A CORRECTED PROOF

We need to close up the discussion regarding the 'logical completeness of the world'⁸⁶ by showing that his world of T -sets does indeed give rise to a topos.

12.1. Theorem. *Badiou's world consisting of T - $\mathcal{S}ets$ —in other words pairs (A, \mathbf{Id}) where $\mathbf{Id} : A \times A \rightarrow T$ satisfies the particular conditions in respect to the complete Heyting algebra structure of T —is 'logically closed', that is, it is an elementary topos. It thus encloses not only pull-backs but also the exponential functor. These make it possible for it to internalize a Badiou's infinity arguments that operate on the power-functor and which can then be expressed from inside the situation despite its existential status.*

Proof. We need to demonstrate that Badiou's world is a topos. Rather than beginning from Badiou's formalism of T -sets, we refer to the standard mathematical literature based on which T -sets can be regarded as sheaves over the particular Grothendieck-topology on the category T : there is a categorical equivalence between T -sets satisfying the 'postulate of materialism' and $\mathcal{S}hvs(T, J)$. The above complications Badiou was caught up with while seeking to 'Platonize' the existence of a topos thus largely go in vain. We only need to show that $\mathcal{S}hvs(T, J)$ is a topos.

Consider the adjoint sheaf functor that always exists for the category of presheaves

$$\mathbf{Id}_\alpha : \mathcal{S}ets^{\mathcal{C}^{op}} \rightarrow \mathcal{S}hvs(\mathcal{C}^{op}, J),$$

where J is the canonical topology. It then amounts to an equivalence of categories. Thus it suffices to replace this category by the one consisting of presheaves $\mathcal{S}ets^{T^{op}}$. This argument works for any category \mathcal{C} rather than the specific category related to an external complete

Heyting algebra T . In the category of $\mathcal{S}ets$ define Y^X as the set of functions $X \rightarrow Y$. Then in the category of presheaves $\mathcal{S}ets^{\mathcal{C}^{op}}$

$$Y^X(U) \cong \text{Hom}(h_U, Y^X) \cong \text{Hom}(h_U \times X, Y),$$

where h_U is the representable sheaf $h_U(V) = \text{Hom}(V, U)$. The adjunction on the right side needs to be shown to exist for all sheaves—not just the representable ones. The proof then follows by an argument based on categorically defined limits whose existence is almost a trivial task⁸⁷. It can also be verified directly that the presheaf Y^X is actually a sheaf.

Finally, for the existence of the subobject-classifier $\Omega_{\mathcal{S}ets^{\mathcal{C}^{op}}}$ ⁸⁸, it can be defined as

$\Omega_{\mathcal{S}ets^{\mathcal{C}^{op}}}(U) \cong \text{Hom}(h_U, \Omega) \cong \{\text{sub-presheaves of } h_U\} \cong \{\text{sieves on } U\}$,
or alternatively, for the category of proper sheaves $\mathcal{S}hvs(\mathcal{C}, J)$, as

$$\Omega_{\mathcal{S}hvs(\mathcal{C}, J)}(U) = \{\text{closed sieves on } U\}.$$

Here it is worth reminding ourselves that the topology on T is defined by a basis $K(p) = \{\Theta \subset T \mid \Sigma\Theta = p\}$. Therefore, in the case of T -sets satisfying the strong 'postulate of materialism', $\Omega(p)$ consists of all sieves S (downward dense subsets) of T bounded by relation $\Sigma S \leq p$. These sieves are further required to be *closed*. A sieve S with an envelope $\Sigma S = s$ is *closed* if for any other $r \leq s$, ie. for all $r \leq s$, one has the implication

$$f_{rs}^*(S) \in J(r) \implies f_{rs} \in S,$$

where $f_{rs} : r \rightarrow s$ is the unique arrow in the poset category. In particular, since $\Sigma S = s$ for the topology whose basis consists of territories on s , we have the equation $1_s^*(S) = f_{ss}^*(S) = S \in J(s)$. Now the condition that the sieve is closed implies $1_s \in S$. This is only possible when S is the maximal sieve on s —namely it consists of *all* arrows $r \rightarrow s$ for $r \leq s$. In such a case it is easily verified that S itself is closed. Therefore, in this particular case

$$\Omega(p) = \{\downarrow(s) \mid s \leq p\} = \{h_s \mid s \leq p\}.$$

It is no more difficult to verify that this is indeed a sheaf whose all amalgamations are 'real' in the sense of Badiou's postulate of materialism. Thus it retains a suitable T -structure.

Let us assume now that we are given an object A , which is basically a functor and thus a T -graded family of subsets $A(p)$. For there to exist a sub-functor $B \hookrightarrow A$ comes down to stating that $B(p) \subset A(p)$ for each $p \in T$. For each $q \leq p$, we also have an injection $B(q) \hookrightarrow B(p)$ compatible (through the subset-representation with respect to A) with the injections $A(q) \hookrightarrow A(p)$. For any given $x \in A(p)$, we can now consider the set

$$\varphi_p(x) = \{q \mid q \leq p \text{ and } x \downarrow q \in B(q)\}.$$

This is a sieve on p because of the compatibility condition for injections, and it is furthermore closed since the map $x \mapsto \Sigma \varphi_p(x)$ is in fact an atom (exercise) and thus retains a real representative $b \in B$. Then it turns out that $\varphi_p(x) = \downarrow(\mathbf{E}b)$. We now possess a transformation of functors $\varphi : A \rightarrow \Omega$ which is natural (diagrammatically compatible). But in such a case we know that $B \hookrightarrow A$ is in turn the pull-back along φ of the arrow true :

$$\begin{array}{ccc} B & \longrightarrow & 1 \\ \downarrow & & \downarrow \text{true} \\ A & \xrightarrow{\varphi} & \Omega. \end{array}$$

It is an easy exercise to show that this map satisfies the universal condition of a pull-back. Indeed, let $\theta : A \rightarrow \Omega$ be another natural transformation making the diagram commute. Given $x \in A(p)$ and $q \rightarrow p$, the pull-back-condition means that $x \downarrow q \in B(q)$ if and only if $\theta(x \downarrow q) = \text{true}_q$, but by naturality of θ this is same as saying $\theta_p(x) \downarrow q = \text{true}_q$. That in turn means $(q \rightarrow p) \in \theta_q(x)$. Therefore, as sets, $\varphi_q(x) = \theta_q(x)$. This concludes our sketch that $\text{true} : 1 \hookrightarrow \Omega$ associating to each singleton of $1(p)$ the maximal sieve $\downarrow(p)$ makes Ω the subobject classifier of the category $\mathcal{S}ets^{T^{op}}$. This is equivalent to the category of T - $\mathcal{S}ets$. \square

13. CONCLUSION—BADIOU INSIDE OUT

In the fourth and the final book of the 'Greater Logic'—the first part of the *Logics of Worlds*—Badiou attempts to conduct the proof demonstrated above. This proof constitutes the 'analytic' of the inquiry 'bearing only on the transcendental laws of being there', or in other words 'the theory of worlds, the elucidation of most abstract

laws of that which constitutes a world qua general form [sic] of appearing'. However, Badiou fails to make the required verifications in order to support the claim for 'generality' that, he believes, denounces the relevance of categorical being(-there)⁸⁹. Instead, Badiou affirms the consequent immersed in his own place to begin with, that is, he adopts a *locally (and logically) bounded*, reductive approach⁹⁰ to topos theory.

If Badiou's dialectics then aims to bridge the 'analytics' (*Logics of Worlds*) with 'dialectics' (*Being and Event*), the phrase 'analytic' refers exactly to the strong, initially *bounding* condition of the constitutive, atomic postulate of materialism. Similar boundaries do not apply to 'weak materialism' and, therefore, the materially weak Grothendieck-topoi could be taken as metaphorical characterizations of Badiou's own, 'material dialectic' philosophy. Grothendieck, early on, approached the mathematical *a priori* from a more synthetic, read Kantian perspective (as a variety of sieves and fibres) whereas Badiou assumes such hierarchies to be initial and dominant, all-encompassing gradings of transcendence.

The postulate of materialism then proves out to be a split postulate, and thus its meaning is very different in non-split situations. Only in the *quasi-split*⁹¹ situation of a locale do the two conditions of materiality coalesce. This in turn seems to contradict with Badiou's other maxim: if dialectics understands 'truths as exceptions', these exceptions need to be understood in terms of the 'most abstract laws'. This is because the 'objective domain of their emergence [...] cannot yet attain the comprehension of the terms that singularize materialist dialectic [...] and subjects as the active forms of these exceptions'⁹².

Badiou's struggle with 'diagrammatics' thus makes his own understanding of these 'abstract laws' flawed precisely because they can, as opposed to what Badiou claims, be thought *mathematically*. Indeed, Badiou's own understanding of the materialist truth becomes *mathematically exceptional* and *split*: it is based either on the primacy of the split concept of truth (the axiom of choice in the *Being and Event*) or the *quasi-split* materialization of force as demonstrated by his fundamental *law of the subject*⁹³. It is Badiou who forces us to choose: 'mathematics or poem' and thus assumes the question of mathematics to split world's domains of appearance into two.

At the same time, Badiou fails to attest his own dialectic maxim, for there is no reason to exempt mathematics from his rule of exceptions: that '[t]here are bodies and languages, *except* that there are truths'⁹⁴. This applies to the world of mathematics. In particular, its embodiment in set theory is exceptional not only because it bounds its truth logically—as a split, two-valued choice characterizing a global hierarchy—but because there is only, and essentially, one instance of such bodies. There are no 'bodies and languages' but, as the name of Badiou's second endeavour suggests, the 'worlds' are multiple only insofar as their 'logics' or languages differ. The materiality of mathematics in this sense appears to be 'democratic' only from within Badiou's inside-out reading.

To Badiou, there is then but a single, 'affirmative proposition'⁹⁵ that declares the 'existence of a pure form of the multiple'. There is no interaction in between; the event may never occur as things happen—in the midst of mathematics as it emerges and takes place in the middle of worlds. Badiou's theory is exceptional precisely because it deals with *an* event whose 'existence = individual = body'⁹⁶: something to occur only in singular; something whose *communication*, whatever its essence, is bounded by formal language similarly to a situation *internal* to an elementary topos but not from the outset.

Were the above-mentioned mistakes then accidental, based on an unfortunate, restricted literature⁹⁷? Was it by chance that Badiou encounters phenomenology only in a way constrained by 'intuitionist logic' internal to Heyting algebras? At least, Badiou himself hardly counts as an 'intuitionist', for in the *Being and Event* he announces that 'that intuitionism has mistaken the route in trying to apply back onto ontology criteria of connection which *come from elsewhere*, and especially from a doctrine of mentally effective operations'. He even declares that 'intuitionism is a prisoner of the empiricist and illusory representation of mathematical objects'⁹⁸. But the reduction to local topoi, bounded by their dialectic languages or 'logics', could serve another, more philosophical purpose.

Indeed, Badiou⁹⁹ consequentially claims to be able to contribute to several philosophical debates. The first problem he discusses has been haunting philosophy since Democritus and Lucretius, who 'had glimpsed the possibility of an infinite plurality of worlds'. Especially

this concerns the capacity of set theory to reflect upon infinity. As Badiou¹⁰⁰ continues, to Aristotle the question of the 'arrangement of the world [which] is essentially finite'—a problem to persist 'at the heart of the Kantian dialectic'.

But the infinity-arguments can be studied from inside any elementary topos, not just *Sets*. There is thus an infinite number of places to make an argument about infinity. The two meanings of infinity—Badiou refers to them as 'bodies' and 'languages'—hardly communicate when it comes to Badiou's own argument. If Badiou was on the right track then while seeking to demonstrate that the logics of worlds are, so to say, infinite. Where he fails is in assuming only logics, not bodies, to be *mathematically infinite*. Any mathematical bodies he discussed are constrained *in* the finite, while makes a case that bodies should only exist as something non-mathematical: that mathematics itself would not count as a body. Therefore, after all, Badiou calls for a return to Aristotle.

But there is also a second concern this train of thought illustrates—one related to Kant indeed. Badiou¹⁰¹ argues that an 'object is nothing but the legislation of appearing'. Instead of propagating a relationalist view—as we have done throughout our categorical discussion—Badiou rather claims that '[t]he definition of a relation must be strictly dependent on that of objects'. This is a claim that, without any reference to an empiricist conclusion (Badiou's experience of mathematics), supports the assumption that the purpose of Badiou's mistake is something more than a mere accident.

'Wittgenstein', he indeed continues, 'having defined the 'state of affairs' as a 'combination of objects', posits that, 'if a thing can occur in a state of affairs, the possibility of the state of affairs must be written into the thing itself'. Badiou still treats the body or the topos of mathematics as such a 'thing': as an object that the 'logics of worlds' are written *into* instead of seeing the things or objects themselves as they occur and occupy mathematics, that is, in relation *for* themselves. Mathematics is still viewed as a stable world—as one 'fixed in its mirage' instead of allowing mathematics itself to happen and occupy its own 'states' in between. Its domain is still incorporeal. Language is yet to discover the process of its own *inscription*.

What is mathematics beyond the 'interior of the world'¹⁰² then? How does it inscribe its own exteriority from inside of its own topos—to gain a grasp of its own alternatives? Badiou could argue that, even if we have shown that there are not only many languages but bodies (topoi) to mathematics, the situations internal to them would still be mute and bounded by formal languages like Wittgenstein anticipated. They would still behave 'logically' if only *from within*. This is to say that there still exists a language, or an amalgam of such languages, that a topos then localizes or embodies without violating any of Badiou's 'democratic materialist' claims about mathematics. Even if as such different from set theory, an elementary topos is thus a world where mathematics happens and 'communicates' itself largely in terms of set theory. The 'same mathematics' still continues, but only from the point of view of a single topos of appearance: not between them.

Instead, when we look at those topoi as they themselves geometrically interact, something occurs that suggests that they are not materially logical or bounded, even if for a specific topos (without a categorical insight) we would be unable to establish this 'immateriality' from within. Badiou's mistake thus comes down to precisely this singular perspective-mistake: he only cares what happens from inside a single situation. Again, by an argument based on internal conditions of such a topos, he would then affirm the consequent by suggesting that no other way for situations to exist were possible: that existence as such would necessarily be understandable only from the perspective of subjective inwardness—the philosophical 'leitmotiv' running from Descartes through Hume to Hegel¹⁰³. He would then overlook how two topoi relate to each other (through the so called geometric morphisms).

But this is not the only way to contest formal language that still appears to regulate elementary topoi if only from within. Instead, what if we could embody structures that did not inscribe a language into its 'thing'—not even internally? Could there be such bodies for which the possibility of whose state of affairs were not 'written into the thing itself', but that could internalize the process of their own interactions? The answer is positive: the world does authorize such forms. Unfortunately, we cannot demonstrate this within the limits of this paper.

We are safe to conclude, however, that the way in which elementary theory displaces the question of materiality does matter. Topos theory counts as an 'expansive construction of the multiple to the multiple-beings'¹⁰⁴ and, like local theory, elementary theory is not exempt from Badiou's material dialectic rule: the category of all *Top* is not itself a topos, and thus the *theory* of such topoi is an expansive construction. It is this theory that is not written into the thing itself insofar as its things are topoi. Something truthful resists the existential seal of any particular topos similarly as Russell encountered the fact that the class of all sets is not itself a set. Unlike Badiou's local theory, which still has only one, single body *Loc*, elementary theory does incorporate an infinity of not only languages but bodies. Badiou's 'logics of worlds', therefore, is only an exception amongst those 'who exist in the world'¹⁰⁵.

In conclusion, if Badiou defines the event negatively in respect to mathematical regulation, he is unable to grasp how that regulation itself happens, for otherwise he would need to situate his own event along with others'. The categorical process, by contrast, appears to falsify at least one of Badiou's two arguments: (1) that the event cannot be *formally* approached or localized except by regulating its *consequences*; or (2) that *formal logic is the only meta-structure*, the only foundation to regulate and intervene truth. Any one of these two propositions can survive only given the failure of the other.

Indeed, let us assume the second postulate and treat category theory as something non-mathematical. This comes down to saying that mathematically it does not exist: it fails to fulfill Badiou's test of 'the true in the Alternative'¹⁰⁶—his *second* test of a body. This would make it 'poetic', something external to the 'Platonic' plane of causes (or 'plane of consistence' as Deleuze and Guattari discuss¹⁰⁷) and, at the same time, category theory itself would appear as an event. It would be approached formally but not in the sense of regulating its consequences, which belong to the domain of logic and causation. Rather, the formal or 'pure' themselves would undergo change, that is, they would happen.

This suffices to demonstrate that the two statements cannot survive in tandem. However, to understand this from the point of view of the first statement, let us note that topos theory *is* a formal approach to the event—to the 'inconsistent' real being, as Badiou regards it.

Elementary theory regulates embodies the event not only in the sense of the locally reconcilable 'generic' instance (φ) as is the case in Cohen's argument. In the precise, *elementary* sense, a topos is an amalgam of mutually inconsistent contexts. Therefore, because topos theory is a formal approach, the first statement suggests that it does *regulate its consequences*: in other words, topos theory is *consequential* to the domain of formal regulation. This appears to contradict with Badiou's second thesis, which argues topos theory, and category theory more broadly, to be inconsequential from the point of view of set theory—the 'plane of consistence'. Otherwise they would make the unmathematical formally manifest, again contradicting with the first thesis.

14. ON MATHEMATICAL STUDIES OF SCIENCE

There is no doubt about the originality of Badiou's attempt to bridge the gap between the philosophical and scientific discourses of materialism. Not only the philosopher Badiou but Badiou as a student of mathematics opened up a whole new box of Pandora: a new way to approach science studies. He has a new style—a style 'in a great writer' which, like Gilles Deleuze's¹⁰⁸ says, 'is always a style of life'.

He said he did not care epistemology, but it is this style that makes him know. But much remains to be said and done. As topos theory embodies change as it appears from inside of mathematics, we need entirely new ways to think about 'being' as subject to the event, the world, and to address its own history and becoming. We need to bring that style to mathematics: this invention of 'a possibility of life, a way of existing'.

That history of this becoming of being runs not only through science: it is visible in philosophy, as it runs through Descartes, Leibniz, Spinoza, Hegel and Heidegger. Yet in many ways we live in an age when progress is unclear. Few people today engage reason deep down their hearts.

NOTES

¹Badiou, Alain (2006), *Being and Event*. Transl. O. Feltman. London, New York: Continuum. [Originally published in 1988.]

²Cohen, Paul J. (1963), 'The Independence of the Continuum Hypothesis', *Proc. Natl. Acad. Sci. USA* 50(6). pp. 1143–1148.

Cohen, Paul J. (1964), 'The Independence of the Continuum Hypothesis II', *Proc. Natl. Acad. Sci. USA* 51(1). pp. 105–110.

³Badiou, Alain (2009). *Logics of Worlds. Being and Event, 2*. Transl. Alberto Toscano. London and New York: Continuum. [Originally published in 2006.]

⁴Badiou, 2009: 540.

⁵See Krömer, Ralf (2007), *Tool and Object: A History and Philosophy of Category Theory*. Science Networks. Historical Studies 32. Berlin: Birkhäuser.

⁶See for example

Awodey, S. (1996), 'Structure in Mathematics and Logic: A Categorical Perspective'. *Philosophia Mathematica* 4 (3). pp. 209–237. doi: 10.1093/phimat/4.3.209.

Palmgren, E. (2009), 'Category theory and structuralism'. url: www2.math.uu.se/~palmgren/CTS-fulltext.pdf, accessed Jan 1st, 2013.

Shapiro, S. (1996), 'Mathematical structuralism'. *Philosophia Mathematica* 4(2), pp. 81–82.

Shapiro, S. (2005) 'Categories, structures, and the Frege-Hilbert controversy: The status of meta-mathematics'. *Philosophia Mathematica* 13(1), pp. 61–62.

⁷Badiou, *Logics of Worlds*, 2009. p. 334.

⁸Badiou, *Logics of Worlds*, 2009. p. 197.

⁹Badiou 2005, 12.

¹⁰Johnstone, Peter T. (1977), *Topos Theory*. London: Academic Press.

Johnstone, Peter T. (2002). *Sketches of an Elephant. A Topos Theory Compendium*. Volume 1. Oxford: Clarendon Press.

¹¹ Mac Lane, Saunders & Ieke Moerdijk (1992), *Sheaves in Geometry and Logic. A First Introduction to Topos Theory*. New York: Springer-Verlag.

¹²Kant, Immanuel (1855), *Critique of Pure Reason*. Trans. J. M. D. Meiklejohn. London: Henry G. Bohn.

¹³Badiou, *Being and Event*, 2005, 4.

¹⁴To demonstrate, if $e \in e$, then $y \in \{e\}$ implies $y = e$ so that one necessarily has $e \in y$ and $e \in \cap\{e\}$ which implies $e \in y \cap \{e\}$. This contradicts the axiom of foundation.

¹⁵Badiou, *Logics of Worlds*, 2009: 39

¹⁶Fraser, Mariam, Kember, Sarah & Lury, Celia (2005), 'Inventive Life. Approaches to the New Vitalism'. *Theory, Culture & Society* 22(1): 1–14.

¹⁷Madarasz, Norman (2005), 'On Alain Badiou's Treatment of Category Theory in View of a Transitory Ontology', In Gabriel Riera (ed), *Alain Badiou—Philosophy and its Conditions*. New York: University of New York Press. pp. 23–44.

¹⁸Badiou, *Logics of Worlds*, 2009. p. 39.

¹⁹Ibid. p. 104.

²⁰Ibid. p. 535.

²¹Ibid. p. 536.

²²Furthermore, there is the associative operator $\text{Hom}(A, B) \times \text{Hom}(B, C) \rightarrow \text{Hom}(A, C)$ that connects any suitable pair of arrows.

²³Badiou, *Being and Event*, 2005. p. 12

²⁴In mathematical logic these structures are usually called Ω -sets, but we try to avoid a confusion between the internal, categorical object Ω and the extensive grading of a locale resulting from a push-forward $T = \gamma_*(\Omega)$ in a bounded morphism $\gamma : \mathcal{C} \rightarrow \mathcal{S}ets$.

²⁵Badiou, *Logics of Worlds*, 2009. pp. 102, 153–155.

²⁶See p. 50, ft. 74.

²⁷Johnstone, *Topos Theory*, 1977, pp. 147–148.

²⁸Badiou, *Logics of Worlds*, 2009. pp. 183–188.

²⁹Eg. Johnstone, *Topos Theory*, 1977, 141.

³⁰Badiou, *Logics of Worlds*, 2009. p. 231.

³¹Ibid., 205.

³²Johnstone, *Topos Theory*, 1977, pp. 39–40.

³³Badiou, *Logics of Worlds*, 2009. p. 248.

³⁴Ibid., 221.

³⁵ In purely categorical terms, an atom is an arrow $X \rightarrow \Omega$ mapping an element $U \rightarrow X$ into $U \rightarrow X \rightarrow \Omega$, thus an element of Ω^X . But arrows $X \rightarrow \Omega$, given that Ω is the sub-object classifier, correspond to sub-objects of X . This localic characterization of a 'sub-object' is thus topos-theoretically justified. The 'element' corresponding with a singleton is exactly the monic arrow from the sub-object to the object. Only in Badiou's localic case the order relation \leq extends to a partial order of the elements of A themselves, and a closed subobject turns out to be the one that is exactly generated by its smallest upper boundary, its envelope as the proof of the existence of the transcendental functor demonstrates. In other words, an atom in Badiou's sense is an atomic subobject of the topos of T -sets. In the diagrammatic terms of general topos theory it agrees with the so called singleton map $\{ \} : X \rightarrow \Omega^X$, which is the exponential transpose of the characteristic map $X \times X \rightarrow \Omega$ of the diagonal $X \hookrightarrow X \times X$.

Expressed in terms of the so called internal Mitchell–Bénabou-language, one may translate the equality $x =_A y = \mathbf{Id}_A(x, y)$ into a different statement $x \in_A \{y\}$ which gives some insight into why such an 'atom' should be regarded as a singleton. Similarly, if a only satisfies the first axiom, the statement $x \in A_a$ would be semantically interpreted to retain the truth-value $a(x)$. Atoms relate to another interesting aspect: the atomic subobjects correspond to arrows $X \rightarrow \Omega$, which allows one to interpret Ω^X as the power-object. Functorially, $\mathbf{P} : \mathcal{E}^{op} \rightarrow \mathcal{E}$ takes the object X to $\mathbf{P}X = \Omega^X$ and an arrow $f : X \rightarrow Y$ maps to the exponential transpose $\mathbf{P}f : \Omega^Y \rightarrow \Omega^X$ of the composite

$$\Omega^Y \times X \xrightarrow{1 \times f} \Omega^Y \times Y \xrightarrow{ev} \Omega,$$

where ev is the counit of the exponential adjunction. The power-functor is sometimes axiomatised independently, but its existence follows from the axioms of finite limits and the subobject-classifier reflecting the fact that these two aspects are separate supposition regardless of their combined positing in the set-theoretic tradition.

³⁶Ibid. p. 104.

³⁷ Wyler, Oswald (1991), *Lecture Notes on Topoi and Quasi-Topoi*. Singapore, New Jersey, London, Hong Kong: World Scientific. , 263.

³⁸Badiou, *Logics of Worlds*, 2009. p. 221.

³⁹For example

Bell, J. L. (1988), *Toposes and Local Set Theories: An Introduction*. Oxford: Oxford University Press.

Borceux, Francis (1994), *Handbook of Categorical Algebra. Basic Theory. Vol. I*. Cambridge: Cambridge University Press.

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Wyler, Oswald (1991), *Lecture Notes on Topoi and Quasi-Topoi*. Singapore, New Jersey, London, Hong Kong: World Scientific.

⁴⁰Badiou, *Logics of Worlds*, 2009. p. 247.

⁴¹Ibid., 271–272.

⁴²Ibid. p. 273.

⁴³Ibid. p. 258.

⁴⁴Ibid., 273–274.

⁴⁵Ibid. p. 263.

⁴⁶See ibid. p. 264.

⁴⁷Ibid., 265–266.

⁴⁸Ibid. p. 278.

⁴⁹Its objects are the elements $\text{Ob}(\mathcal{C}_T) = \mathbf{PT}$. Now for $p, q \in T$ define $\text{Hom}_{\mathcal{C}_T}(p, q) = \{\leq\}$ if $p \leq q$ and $\text{Hom}_{\mathcal{C}_T}(p, q) = \emptyset$ otherwise.

⁵⁰Badiou, *Logics of Worlds*, 2009. pp. 225–226.

⁵¹Johnstone, *Topos Theory*, 1977, pp. 10–11.

⁵²The construction of the sheaf-functor proceeds by defining

$$P^+ = \varinjlim_{R \in J(C)} \text{Match}(R, P), \quad P \mapsto P^{++},$$

where $\text{Match}(R, P)$ denotes the matching families. When applied twice, it amounts to a left-adjoint

$$a : \mathcal{S}ets^{\mathcal{C}^{op}} \rightarrow \mathcal{S}hvs(\mathcal{C}, J)$$

which sends a sheaf to itself in the category of presheaves.

⁵³Mac Lane & Moerdijk, *Sheaves in Geometry*, 1992, 277.

⁵⁴Higher categories are much more 'Grothendieckian' than are the Grothendieck-topoi.

⁵⁵Johnstone, *Topos Theory*, 1977, p. xii.

⁵⁶If λ is a property, then the set discerned by this property actually exists. In set-theoretical formalism it can be expressed as

$$\forall w_1, \dots, w_n \forall A \exists B \forall X (x \in B \iff [x \in A \wedge \lambda(x, w_1, \dots, w_n, A)]),$$

where x, w_1, \dots, w_n are free variables of the statement λ .

⁵⁷Ibid., xiii.

⁵⁸Ibid., 39.

⁵⁹Badiou, *Being and Event*, 2006, p. 401.

⁶⁰Johnstone, *Topos Theory*, 1977, p. 145.

⁶¹Ibid., 146.

⁶²Mac Lane & Moerdijk 1992, 141, Lemma 1.

⁶³Badiou, *Logics of Worlds*, 2009. pp. 289–295

⁶⁴In fact, there is an implicit definition of a Grothendieck-topology involved already when Badiou expresses the Cohen’s procedure in the *Being and Event*

⁶⁵Ibid. p. 291

⁶⁶ P is a sheaf, if and only if for every covering sieve S on C the inclusion $S \hookrightarrow h_C$ into the presheaf $h_C : D \mapsto \text{Hom}(D, C)$ induces an isomorphism $\text{Hom}(S, P) \cong \text{Hom}(h_C, P)$. In categorical terms, a presheaf P is a sheaf for a topology J , if and only if for any cover $\{f_i : C_i \rightarrow C \mid i \in I\} \in K(C)$ in the basis K the diagram

$$P(C) \rightarrow \prod_i P(C_i) \rightrightarrows \prod_{i,j \in I} P(C_i \times_C C_j)$$

is an equaliser of sets (where the ‘fibre-product’ on the right hand side is the canonical pull-back of f_i and f_j (Mac Lane & Moerdijk 1992, 123). Furthermore, a topology J is called *subcanonical* if all representable presheaves, that is, all functors $h_C : D \mapsto \text{Hom}(D, C)$ are sheaves on J .

⁶⁷More precisely, when working over the 2-category of elementary topoi \mathcal{Top} , if $f : \mathcal{E} \rightarrow \mathcal{F}$ is a geometric morphism which is bounded, there is an inclusion $\mathcal{E} \hookrightarrow \mathcal{F}^{\mathbf{C}}$, where \mathbf{C} is an ‘internal category’ of \mathcal{F} . This follows from the so called Giraud–Mitchell–Diaconescu-theorem. Unless the geometric morphism $\gamma : \mathcal{E} \rightarrow \mathcal{S}ets$ is bounded, the $\gamma^*(\Omega_{\mathcal{E}})$ does not need to be a complete Heyting algebra. Only in the bounded, ‘strong’ case $\gamma_*(\Omega_{\mathcal{E}})$ is an external complete Heyting algebra and it is then the full subcategory of open objects in \mathcal{E} . Ibid., 150.

⁶⁸A terminal object 1 means that for every object X there is a *unique* arrow $X \rightarrow 1$, every arrow can be extended to terminate at 1 . To the notion of a pull-back (or fibered product) we will come to shortly.

⁶⁹Generally a functor between categories $f : \mathcal{F} \rightarrow \mathcal{G}$ maps each object $X \in \mathcal{F}$ to an object $f(X) \in \mathcal{G}$ and each arrow $a : X \rightarrow Y$ to an arrow $f(a) : f(X) \rightarrow f(Y)$ ‘naturally’ in the sense that $f(a \circ b) = f(a) \circ f(b)$. Therefore, f induces a map $f : \text{Hom}_{\mathcal{F}}(X, Y) \rightarrow \text{Hom}_{\mathcal{G}}(f(X), f(Y))$. If $g : \mathcal{G} \rightarrow \mathcal{F}$ is another functor, then $g \circ f : \mathcal{E} \rightarrow \mathcal{E}$ and $f \circ g : \mathcal{E} \rightarrow \mathcal{E}$ are two functors. It is said that another functor $g : \mathcal{G} \rightarrow \mathcal{F}$ is right (resp. left) adjoint of f , notated by $f \dashv g$, if given any pair of objects $X \in \mathcal{E}$ and $Y' \in \mathcal{G}$ there is also a ‘natural’ isomorphism $\Phi_{X, Y'} : \text{Hom}_{\mathcal{F}}(X, g(Y')) \xrightarrow{\sim} \text{Hom}_{\mathcal{G}}(f(X), Y')$.

⁷⁰In this particular case, the pull-back-condition means that, given any $\theta : Z \rightarrow X$ so that $\varphi_{\sigma} \circ \theta : Z \rightarrow \Omega$ is the (unique) arrow $Z \longrightarrow 1 \xrightarrow{\text{true}} \Omega$, then there is a unique map $\gamma : Z \rightarrow Y$ so that $\sigma \circ \gamma = \theta$.

⁷¹For the definition and further implications see Johnstone, *Topos Theory*, 1977, *Topos Theory*, pp. 16–17.

⁷²Badiou, *Logics of Worlds*, 2009. pp. 338–339.

⁷³Badiou, *Logics of Worlds*, 2009. p. 339.

⁷⁴Despite the fact that the category of *Sets*—which isn't itself a set as Badiou demonstrates in the case of the 'world **m**'—is 'paradoxical' based on Russell's argument, one may now safely replace the category *Sets* with any category \mathcal{S} , which satisfies the basic properties of the set-theoretic world *Sets*: a topos satisfying the categorical version of the axiom of choice (AC) and support generators (SG) and has a natural number object; and finally its subobject-classifier equals **2** or is 'bi-valued'.

⁷⁵Foucault, Michel (1995), *Discipline and Punish: The Birth of the Prison*, trans. Alan Sheridan. New York: Random House. pp. 171, 205.

⁷⁶Ibid., 313.

⁷⁷Ibid., 317

⁷⁸Ibid., 315–316.

⁷⁹If a category \mathcal{J} is finite (in the number of objects and arrows), we can consider the category $\mathcal{E}^{\mathcal{J}}$ whose objects are functors $F : \mathcal{J} \rightarrow \mathcal{E}$ and arrows so called 'natural transformations'. There is a diagonal functor $\Delta_{\mathcal{J}} : \mathcal{E} \rightarrow \mathcal{E}^{\mathcal{J}}$ defined as the constant functor associating $\Delta(X)(j) = X$ for all $j \in \mathcal{J}$. Then the limit-condition means that there is a right adjoint to $\Delta_{\mathcal{J}}$ which is $\varprojlim_{\mathcal{J}} : \mathcal{E}^{\mathcal{J}} \rightarrow \mathcal{E}$ and designates a limit of \mathcal{J} -indexed objects. An alternative definition supposes only the existence of pull-backs and a terminal-objects from which the existence of all finite limits follows. Those are limits of the categories of the empty category (terminal object) and the category with only one object and two nonidentity morphisms $\rightarrow \bullet \leftarrow$ (pull-back).

⁸⁰Badiou, *Logics of Worlds*, 2009. p. 345.

⁸¹Ibid., lemma 2, 346.

⁸²Ibid., lemma 3, 346.

⁸³Ibid., lemma 4, 347.

⁸⁴Ibid., 347–348.

⁸⁵Ibid., 350–352.

⁸⁶Ibid., 318.

⁸⁷Johnstone, *Topos Theory*, 1977, pp. 24–25.

⁸⁸Ibid., p. 25.

⁸⁹Badiou, *Logics of Worlds*, 2009. p. 299.

⁹⁰I showed how he reduces topos theory to a theory of so called locales—elementary topoi that are defined over the category of *Sets* in such a way that the geometric morphism $\gamma : \mathcal{E} \rightarrow \mathcal{S}$ is *bounded* and *logical*. This makes Badiou to falsely identify the 'internal' logic of a topos with the logic which emerges on the set-theoretic surface of $T := \gamma_*(\Omega)$ because for a general *Sets*-topos (eg. Grothendieck-topos), these two logics do not agree.

⁹¹The split understanding of truth refers to the condition of the category of *Sets* in which the subobject-classifier Ω is the split set consisting of only two elements: true and false. Similarly, although the truth in a local doesn't split in this explicit manner, it is quasi-split inasmuch as the T anyway set-theoretically represents truth as extensive 'values' or 'degrees'. As a remark, Badiou misleadingly regards them as 'degrees of intensity' inasmuch as the categorical designation of a topos

expresses such 'degrees' in a much more intensive meaning. In contrast, Badiou's set-theoretic representation makes these 'degrees of truth' *extensive* when they are set-theoretically incorporated.

⁹²Ibid. p. 299.

⁹³Badiou, *Being and Event*, 2006. p. 401.

⁹⁴Badiou, 2009, 6.

⁹⁵Ibid., 249.

⁹⁶Ibid, 2.

⁹⁷ In the mathematical references Badiou makes, everything seems to relate only to such logically bounded, localic structures. Badiou (ibid. p. 538) mentions such books as:

Bell, *Toposes and Local Set Theories: An Introduction*, 1988.

Borceux, *Handbook of Categorical Algebra. Basic Theory. Vol. I.*, 1994.

Goldblatt, *The Categorical Analysis of Logic*, 1984.

Wyler, *Lecture Notes on Topoi and Quasi-Topoi*, 1991.

⁹⁸Ibid., 249.

⁹⁹Badiou, *Logics of Worlds*, 2009. p. 300.

¹⁰⁰Ibid. p. 300

¹⁰¹Ibid. p. 301

¹⁰²Ibid. p. 262.

¹⁰³Badiou, *Logics of Worlds*, 2009. p. 411.

¹⁰⁴Ibid. p. 309.

¹⁰⁵See ibid. p. 310.

¹⁰⁶Badiou, *Logics of Worlds*, 2009, 430.

¹⁰⁷Deleuze, Gilles & Félix Guattari (1988), *Thousand Plateaus—Capitalism and Schizophrenia*. Transl. Brian Massumi, Minneapolis, University of Minnesota Press. [Originally published in 1980.]

¹⁰⁸Deleuze, Gilles (1990), *Negotiations, 1972–1990*. Trans. Martin Joughin. New York: Columbia University Press. p. 100.

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